Cooperation and allocation

Hendrickx, R.L.P.

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ter verkrijging van de graad van doctor aan de Universiteit van Tilburg, op gezag van de rector magnificus, prof.dr. F.A. van der Duyn Schouten, in het openbaar te verdedigen ten overstaan van een door het college voor promoties aangewezen commissie in de aula van de Universiteit op vrijdag 4 juni 2004 om 14.15 uur door

Ruud Leo Peter Hendrickx

geboren op 7 september 1977 te Venlo.

Promotores: prof.dr. P.E.M. Borm
prof.dr. S.H. Tijs

Festina lente
cauta fac omnia mente

Spooj dich langzaam
en gebroèk diene kieëbus

Wapenspreuk van Venlo

Preface

My career as researcher at Tilburg University started back in 1998, when as a second year student of Econometrics (without OR at that time) I became research assistant to Ton van Schaik. For 18 months, we tried to measure the rather vague concept of social capital by means of principal component analysis and used this to perform an empirical study of the effect of social capital on key macro-economic variables like growth and unemployment. As this thesis attests, this strand of research did not turn out to be entirely my cup of tea, but the critical attitude and, above all, the enthusiasm Ton exhibited for discovering the unknown taught and inspired me a lot. For this I deeply thank him.

The seeds of my second stab at research were sown in the autumn of 1998, when I followed a course in game theory by Stef Tijs. I remember vividly how our class was intimidated by the chaotic nature and sheer number of Stef's slides. Without a doubt, his unorthodox and intriguing method of teaching has contributed greatly to my subsequent interest in the field of game theory. The third year thesis (now called Bachelor's thesis) I started writing in the following spring – which deals with correlated equilibria in bimatrix games – has laid the foundation for my next five years in Tilburg.

Whereas Stef was responsible for enthusing me for game theory, it was Peter Borm who made the effort to persuade me to become a PhD student in Tilburg. His and Stef's trademark cooperative approach to doing research has resulted in four years of fruitful and, more importantly, pleasant research. For this, I am much indebted to my two promotores.

I would also like to thank the other committee members, Michael Maschler, Carles Rafels, Hans Reijnierse, Dolf Talman and Judith Timmer, for taking the time to read and evaluate this thesis.

Of course, doing research is not only about reading and writing articles. Game theory and game practice are, or at least should be, Siamese twins. To compensate for the lack of noncooperative game theory in this thesis, I have spent much of my time in Tilburg practicing this discipline, in particular zero-sum games. I enjoyed many evenings and lunch breaks playing (and far too often, losing) a wide variety of games. Alex, Arantza, Bas, Frans, Hendri, Jacco, Johan, Marcel, Marieke, Paul, Ramon and Stefan, thanks for this. I would also like to thank various members of De Wolstad and the wider chess community for revealing my inaptitude at this most frustrating of occupations.

A valuable contribution to this thesis was made by my two officemates during the past four years, Grzegorz (pronounced *Greg*) and Marcel. Their diversions during working hours (some of them serious, some not-so-serious) certainly helped keeping the show on the road. I am particularly grateful to Marcel for our moral, linguistic, psychological, TEXnical, didactic, philosophical and, most of the time, pointless discussions on the state of the world.

I was brought up with the notion that one's family should always be central in one's life. Fortunately, mine are. Without the support and warmth of my family, my life would be poorer and I thank them for keeping things in (a joyful) perspective. My parents are of course the main contributors to keeping me on the straight and narrow.

Finally, I am much indebted to Marloes and Coen for agreeing to be my *paranimfen.*

Neem deel aan het onderwijs en jullie zullen
daardoor veel zilver en goud verwerven.
Sir 51,28

Het onderzoeken van moeilijke dingen is eervol.
Spr 25,27

Wat is het leven voor iemand die geen wijn heeft?
Sir 31,27

Contents

Chapter 1

Introduction

Game theory deals with models of competition and cooperation. Since the appearance of Von Neumann and Morgenstern (1944), many game theorists have tried to capture economic behaviour and other situations in which agents (or rather, players) interact in formal models, with the purpose of analysing them in a coherent and systematic way.

The competitive nature of interaction is the topic of *noncooperative game theory*. There, players are considered as individual utility-maximisers playing a game against each other. The term *game* in this context is interpreted as any interactive situation in which a player's payoff depends not only on his own choice of actions, but also on the actions of his opponents. One can think of parlour games (eg, chess) or more worldly games like firms competing in an oligopolistic market. The main focus in noncooperative game theory is on formalising notions of rationality, the main one being the concept of equilibrium.

In *cooperative game theory*, which is the subject matter of this thesis, cooperation between the players is studied. By working together in coalitions, players can generate benefits. A typical example is that of a number of firms cooperating in order to save costs. Not only is it interesting to know how players can cooperate in an optimal way, but also the problem of *allocation* arises. The central question in cooperative game theory is how the proceeds of cooperation can or should be divided among the players in a fair way. To assess this, one has to come up with properties on the basis of which allocation rules can be compared.

Cooperative game theory comprises many different models. By far the most popular of these is the model of *transferable utility games*. One can think of a transferable utility game as an allocation problem in which an amount of money is to be divided and where one abstracts from the fact that the players involved might put different value on the monetary payoffs they may receive. Transferable utility games were already introduced in Von Neumann and Morgenstern (1944) and have since formed the main pillar of cooperative game theory.

The second main model in cooperative game theory is that of *nontransferable utility games*, introduced by Aumann and Peleg (1960). Such a game arises when the objects to be divided are not valued in the same way by all the players. As one might imagine, such situations are much harder to analyse than transferable utility games. Eg, the well-known characterisation of nonemptiness of the core of a transferable utility game in terms of balancedness by Bondareva (1963) has only recently been extended to the context of nontransferable utility games by Predtetchinski and Herings (2003).

In the first few chapters of this thesis, we consider some well-known concepts in transferable utility theory and extend them to the context of nontransferable utility. Convexity is the subject of Chapter 3. *Convexity* for transferable utility games was already introduced in Shapley (1971) and has various equivalent definitions (cf. Ichiishi (1981)), each having its own interpretation. The most direct interpretation is in terms of increasing marginal contributions: a game is convex if the marginal contribution of a player to a coalition increases when the coalition that he joins becomes larger. This nice marginalistic interpretation, however, has not been central in the extensions of convexity to nontransferable utility games up till now. Vilkov (1977) and Sharkey (1981) generalise convexity on the basis of its so-called supermodular interpretation, yielding ordinal convexity and cardinal convexity, respectively.

In Chapter 3, we define three new types of convexity for nontransferable utility games that are based on the marginalistic interpretation: coalition merge convexity, individual merge convexity and marginal convexity. The main message of this chapter is that although in the case of transferable utility, all convexity notions boil down to the same, for nontransferable utility they are different. We investigate all the relations between the five types of convexity and consider them in the light of some special classes of games and of some rules.

In both transferable and nontransferable utility theory, various concepts of *monotonicity* have been studied. An interesting contribution in this field is Sprumont (1990), who introduces the concept of *population monotonic allocation scheme*. In a convex transferable utility game, each extended marginal vector is such a scheme and in his concluding remarks, Sprumonts asks the question whether this result can be extended to games with nontransferable utility. In Chapter 4 we answer this question in the affirmative by showing that individual merge convexity is a sufficient condition for each extended marginal vector to constitute a population monotonic allocation scheme.

In the same chapter, we also introduce a new type of monotonicity, *drop out monotonicity*, which we analyse in the context of sequencing situations (cf. Curiel et al. (1989)). A sequencing rule is said to be drop out monotonic if all remaining players become better off if one of them leaves the queue. This natural property is not satisfied by many well-known sequencing rules. We show that, in fact, there is at most one drop out monotonic rule that is stable, ie, always yielding a core element. This so-called μ rule, which is a marginal vector of the corresponding sequencing game, turns out to be drop out monotonic on the simple class of sequencing games with linear cost functions. For many other classes of regular cost functions, no stable drop out monotonic rule exists.

Myerson (1977) introduces a cooperative model in which cooperation between the players is modelled by a communication network as well as a transferable utility game. This underlying game models the benefits that the coalitions can generate if they cooperate, whereas the communication network models the extent to which this cooperation is possible. These two ingredients result in a so-called graph-restricted game, which reflects both the underlying possibilities of the players and the extent to which these can come to fruition. For such *communication situations*, Van den Nouweland and Borm (1991) and Slikker and Van den Nouweland (2001) analyse the problem of inheritance of properties. In short, what conditions must a communication network satisfy so that for every underlying game that satisfies a certain property, the resulting graph-restricted game satisfies the same property? In Chapter 5, we extend this analysis to nontransferable utility games and point out some differences between the two models.

The distinction between cooperative and noncooperative behaviour is not always clear-cut (cf. Van Damme and Furth (2002)). In many economic situations, both elements are present and a unified approach is called for. In a noncooperative model, like the Cournot oligopoly model, one might want to explicitly model the possibility of collusion. Or the other way around, in a cooperative model, one might want to incorporate some strategic elements in order to come to a more realistic or fairer solution.

In Chapter 6, we introduce the class of *spillover games*. This is basically a transferable utility model with an extra ingredient: spillovers. Whenever a coalition of players decides to cooperate, they do not only generate a payoff to themselves, but also to the players outside that coalition. Such spillovers can arise if, eg, the coalition inflicts externalities as a result of pollution. In Chapter 6 we present this new model and extend some well-known concepts for transferable utility games to this new class.

Not only can economic externalities result in spillovers, also in operations research problems, such spillovers can occur. In the case of minimum cost spanning trees, a coalition building a public network obviously influences the possibilities and hence the payoffs of the other players in the game. We analyse this public-private connection problem using our new model, which results in an elegant depiction of the problem of free riding.

As mentioned before, cooperation and allocation are inextricably linked. An allocation problem arises whenever a bundle of goods is held in common by a group of individuals and must be alloted to them individually. The purest allocation problem is a *bankruptcy situation*, as modelled by O'Neill (1982). In a bankruptcy situation, there is a sum of money, the estate, available to be divided among a group of players, each having a single claim on the estate. This simple division problem has inspired many allocation proposals, each having its own appealing properties. Although there has been a recent upsurge in attention for bankruptcy situations (see, eg, the survey article by Thomson (2003)), still a lot has to be explored. Solving this easy problem may and should help us understand more difficult allocation problems.

An interesting variation on the bankruptcy model is provided by Pulido et al. (2002). In their model, the players do not only have a claim on the estate available, but in addition there is also an objective criterion to compute a reference amount for each player. Obviously, this extra information should be used to find a fair allocation of the estate. In Chapter 7, we analyse these *bankruptcy situations with references* and propose a compromise method to divide the estate.

In Chapter 8, we take a broader view on bankruptcy. Instead of each player having a single claim on the estate, we consider the situation in which there are multiple issues on the basis of which the estate is to be divided. These issues are equally valid, so the asymmetry which we can exploit in the case of claims and references is not present here. The resulting model of *multi-issue allocation situations* can be seen as a general framework for division problems, to which, depending on the context, many methods of solution can be applied.

Related to this new type of allocation problem, we define two transferable utility games and obtain the nice theoretical result that this class of games coincides with the class of all nonnegative exact games. We propose two rules, based on the run-to-the-bank rule that was already introduced by O'Neill (1982), where the interdependency between the various issues is reflected by compensation payments.

O'Neill uses a property of *consistency* to characterise the run-to-the-bank rule. In this thesis, we frequently draw on this consistency principle to provide characterisations of run-to-the-bank-like rules. In Chapter 8, we give the first of these.

The main part of Chapter 9 considers a further extension of the run-to-the-bank rule in the context of multi-issue allocation situations, which unlike the extensions in Chapter 8 always yields a core element. This new extension is based on a two-stage approach, where the issues and the players are treated in subsequent order rather than simultaneously. Also this new rule is characterised by a consistency property, called issue-consistency.

Our final model related to bankruptcy is the subject of Chapter 10. In many economic situations where players can cooperate, one can a priori partition the players into groups. These so-called *a priori unions* were first analysed by Owen (1977), who adapted the Shapley value to take these unions into account. Whereas Owen considers a general transferable utility framework with a priori unions, in Chapter 10 we study the more specific context of bankruptcy situations. In a bankruptcy situation, the players can often be partitioned into a priori unions on the basis of the nature of their claim. The main focus of the chapter is on extending the constrained equal award rule for bankruptcy situations to take the union structure into account. We introduce and characterise two such extensions.

Geometry plays a minor, though interesting role in cooperative game theory. Many set-valued solutions for transferable utility games have a nice geometric structure and various rules can be described in terms of geometric principles. The best-known example is the Shapley value, which is the barycentre of the Weber set. In Chapter 11, we characterise an adaptation of the compromise value, the τ^* value, as the barycentre of the edges of the core cover. The proof requires some intricate machinery which provides some insight into the structure of the core cover.

Chapter 2

Preliminaries

2.1 Basic notation

The set of all natural numbers is denoted by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$, the set of nonnegative (nonpositive) reals by $\mathbb{R}_+$ ($\mathbb{R}_-$) and the set of positive (negative) reals by $\mathbb{R}_{++}$ ($\mathbb{R}_{--}$). For a finite set N, we denote its power set, ie, the collection of all its subsets, by 2^N and its number of elements by $|N|$. By $\mathbb{R}^N$ we denote the set of elements of $\mathbb{R}^{|N|}$ whose entries are indexed by N, or equivalently, the set of all real-valued functions on N. An element of $\mathbb{R}^N$ is denoted by a vector $x = (x_i)_{i \in N}$. For $S \subset N, S \neq \emptyset$, we denote the restriction of x on S by $x_S = (x_i)_{i \in S}$. For $x, y \in \mathbb{R}^N$, $y \geq x$ denotes $y_i \geq x_i$ for all $i \in N$, $y > x$ denotes $y_i > x_i$ for all $i \in N$ and $y \gneqq x$ denotes $y \geq x, y \neq x$.

For a finite set N and a subset $S \subset N$, we denote by e^S the vector in $\mathbb{R}^N$ defined by $e_i^S = 1$ for all $i \in S$ and $e_i^S = 0$ for all $i \in N \backslash S$. If $S = \{i\}$, we denote the corresponding unit vector by e^i. By 0^N we denote the zero vector in $\mathbb{R}^N$.

An *ordering* of the elements in N is a bijection $\sigma : \{1, \ldots, |N|\} \to N$, where $\sigma(i)$ denotes which element in N is at position i. The notation $\sigma = (a_1, a_2, \ldots, a_n)$ is used as shorthand for $\sigma(1) = a_1$, $\sigma(2) = a_2$, $\ldots$, $\sigma(n) = a_n$. The set of all $|N|!$ orderings of N is denoted by $\Pi(N)$.

2.2 TU games

A *cooperative game with transferable utility*, or *TU game*, is a pair (N, v), where $N = \{1, \ldots, n\}$ denotes the set of players and $v : 2^N \to \mathbb{R}$ is the *characteristic function*, assigning to every coalition $S \subset N$ of players a value, or *worth*, $v(S)$, representing the total payoff to this coalition of players when they cooperate. By convention, $v(\emptyset) = 0$. We denote the class of all TU games with player set N by TU^N. Where no confusion can arise, we denote a game $(N, v) \in TU^N$ by v.

The *subgame* of (N, v) with respect to coalition $S \subset N, S \neq \emptyset$ is defined as the TU game (S, v^S) with $v^S(T) = v(T)$ for all $T \subset S$.

For a game $v \in TU^N$, the *imputation set* $I(v)$ is defined by

$$I(v) = \{x \in \mathbb{R}^N \mid \sum_{i\in N} x_i = v(N), \forall_{i\in N} : x_i \geq v(\{i\})\}.$$

The *core* $C(v)$ is defined by

$$C(v) = \{x \in \mathbb{R}^N \mid \sum_{i\in N} x_i = v(N), \forall_{S\subset N} : \sum_{i\in S} x_i \geq v(S)\}.$$

A core element is stable in the sense that if such a vector is proposed as allocation for the grand coalition, no coalition will have an incentive to split off and cooperate on their own.

A game is called *balanced* if its core is nonempty and *totally balanced* if the core of each of its subgames is nonempty.

A game $v \in TU^N$ is called *superadditive* if for all coalitions $S, T \subset N$ such that $S \cap T = \emptyset$ we have

$$v(S) + v(T) \leq v(S \cup T).$$

The *marginal vector* $m^\sigma(v)$ of a game $v \in TU^N$ corresponding to the ordering $\sigma \in \Pi(N)$ is defined by

$$m^\sigma_{\sigma(k)}(v) = v(\{\sigma(1), \ldots, \sigma(k)\}) - v(\{\sigma(1), \ldots, \sigma(k-1)\})$$

for all $k \in \{1, \ldots, n\}$.

The *Shapley value* of a game $v \in TU^N$, $\Phi(v)$, (cf. Shapley (1953)) is defined as the average of the marginal vectors

$$\Phi(v) = \frac{1}{n!} \sum_{\sigma\in\Pi(N)} m^\sigma(v).$$

For a game $v \in TU^N$, the *utopia vector* $M(v) \in \mathbb{R}^N$ is defined by

$$M_i(v) = v(N) - v(N\backslash\{i\})$$

for all $i \in N$, and the *minimal right vector* $m(v) \in \mathbb{R}^N$ by

$$m_i(v) = \max_{S:i\in S}\left[v(S) - \sum_{j\in S\setminus\{i\}} M_j(v)\right]$$

for all $i \in N$. A game $v \in TU^N$ is called *compromise admissible* (or *quasi-balanced*) if $m(v) \leq M(v)$ and $\sum_{i\in N} m_i(v) \leq v(N) \leq \sum_{i\in N} M_i(v)$. We denote the set of all compromise admissible games with player set N by CA^N.

For a compromise admissible game the *compromise value* or τ *value* (cf. Tijs (1981)) is defined as the linear combination of the utopia vector and the minimal right vector that is efficient, ie, for all $v \in CA^N$,

$$\tau(v) = \lambda M(v) + (1-\lambda)m(v)$$

with $\lambda \in [0,1]$ such that $\sum_{i\in N} \tau_i(v) = v(N)$.

For a game $v \in TU^N$, the *core cover* is defined by

$$CC(v) = \{x \in \mathbb{R}^N \mid \sum_{i\in N} x_i = v(N), m(v) \leq x \leq M(v)\},$$

so a game is compromise admissible if and only if its core cover is nonempty. Tijs and Lipperts (1982) show that $C(v) \subset CC(v)$, so every balanced game is compromise admissible. A game $v \in TU^N$ is called *semi-convex* (cf. Driessen and Tijs (1985)) if it is superadditive and

$$m_i(v) = v(\{i\})$$

for all $i \in N$.

The *excess* of coalition $S \subset N$ for imputation $x \in I(v)$ is defined by

$$E(S,x) = v(S) - \sum_{i\in S} x_i.$$

If x is proposed as an allocation vector, the excess of S measures to which extent S is satisfied with x: the lower the excess, the more pleased S is with the proposed allocation. The idea behind the nucleolus is to minimise the highest excesses in a hierarchical manner.

Let $x, y \in \mathbb{R}^t$. Then we say that x is lexicographically smaller than or equal to y, or $x \leq_L y$, if $x = y$ or if there exists an $s \in \{1,\ldots,t\}$ such that $x_k = y_k$ for all $k \in \{1,\ldots,s-1\}$ and $x_s < y_s$. For a finite set A, we denote $x \leq_L^* y$ with $x, y \in \mathbb{R}^A$, if $x' \leq_L y'$ where x' (y') is the vector in $\mathbb{R}^{|A|}$ containing the elements of x (y) in decreasing order.

Let $v \in TU^N$ be a game with a nonempty imputation set. The *nucleolus* of v, $\nu(v)$, (cf. Schmeidler (1969) and Maschler et al. (1992)) is the unique point in $I(v)$ for which the excesses are lexicographically minimal, ie,

$$(E(S,\nu(v)))_{S\subset N} \leq_L^* (E(S,x))_{S\subset N}$$

for all $x \in I(v)$.

2.3 NTU games

A *cooperative game with nontransferable utility*, or *NTU game*, is a pair (N,V), where $N = \{1,\ldots,n\}$ is the set of players and V is the payoff map assigning to each coalition $S \subset N, S \neq \emptyset$ a subset $V(S)$ of $\mathbb{R}^S$. This set represents all the payoff vectors that coalition S can obtain when they cooperate.

We impose some conditions on V: for all $i \in N$,

$$V(\{i\}) = (-\infty, 0]$$

and for all $S \subset N, S \neq \emptyset$ we have

$$V(S) \text{ is nonempty and closed,}$$

$$V(S) \text{ is comprehensive, ie, } x \in V(S) \text{ and } y \leq x \text{ imply } y \in V(S),$$

$$V(S) \cap \mathbb{R}^S_+ \text{ is bounded.}$$

Furthermore, we assume that (N,V) is *monotonic*:

$$\forall_{S\subset T\subset N, S\neq\emptyset}\forall_{x\in V(S)}\exists_{y\in V(T)} : y_S \geq x.$$

Note that we do not define $V(\emptyset)$. The class of NTU games with player set N is denoted by NTU^N. Again, we sometimes use V rather than (N,V) to denote an NTU game.

NTU games generalise TU games. Every TU game (N,v) gives rise to an NTU game (N,V) by defining $V(S) = \{x \in \mathbb{R}^S \mid \sum_{i\in S} x_i \leq v(S)\}$ for all $S \subset N, S \neq \emptyset$.

The *subgame* of (N,V) with respect to coalition $S \subset N, S \neq \emptyset$ is defined as the NTU game (S, V^S) with $V^S(T) = V(T)$ for all $T \subset S, T \neq \emptyset$.

The set of *Pareto efficient allocations* for coalition $S \subset N, S \neq \emptyset$, denoted by $Par(S)$, is defined by

$$Par(S) = \{x \in V(S) \mid \neg\exists_{y\in V(S)} : y \gneqq x\},$$

its set of *weak Pareto efficient allocations* $WPar(S)$ is defined by

$$WPar(S) = \{x \in V(S) \,|\, \neg\exists_{y \in V(S)} : y > x\}$$

and its set of *individually rational allocations* is defined by[1]

$$IR(S) = \{x \in V(S) \,|\, \forall_{i \in S} : x_i \geq 0\} = V(S) \cap \mathbb{R}^S_+.$$

The *imputation set* of a game $V \in NTU^N$, denoted by $I(V)$, is defined by

$$I(V) = IR(N) \cap WPar(N).$$

The *core* of an NTU game (N, V) consists of those elements of $V(N)$ for which it holds that no coalition $S \subset N, S \neq \emptyset$ has an incentive to split off:

$$C(V) = \{x \in V(N) \,|\, \forall_{S \subset N, S \neq \emptyset} \neg\exists_{y \in V(S)} : y > x_S\}.$$

Again, we call a game $V \in NTU^N$ *balanced*[2] if it has a nonempty core and *totally balanced* if all its subgames have a nonempty core.

An NTU game $V \in NTU^N$ is called *superadditive* if for all coalitions $S, T \subset N$ such that $S \neq \emptyset, T \neq \emptyset, S \cap T = \emptyset$ we have

$$V(S) \times V(T) \subset V(S \cup T).$$

This definition of superadditivity is a straightforward generalisation of the concept of superadditivity for TU games. In addition, we define a weaker property concerning only the merger between individual players and coalitions rather than between two arbitrary disjoint coalitions. A game $V \in NTU^N$ is called *individually superadditive* if for all $i \in N$ and for all $S \subset N \backslash \{i\}, S \neq \emptyset$ we have

$$V(S) \times V(\{i\}) \subset V(S \cup \{i\}).$$

Note that individual superadditivity is stronger than monotonicity.

The *marginal vector* $M^\sigma(V)$ of a game $V \in NTU^N$ corresponding to the ordering $\sigma \in \Pi(N)$ (cf. Otten et al. (1998)) is defined by

$$M^\sigma_{\sigma(1)}(V) = 0$$

and

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[1] Recall that we assumed zero-normalisation.

[2] In the case of (total) balancedness, we abuse standard terminology. Formally, balancedness is a property of TU games, which in Bondareva (1963) is shown to coincide with nonemptiness of the core. For both TU and NTU games, we refer to the latter property as balancedness.

$$\begin{aligned} M^{\sigma}_{\sigma(k)}(V) = \quad \max\{x_{\sigma(k)} \mid \quad & x \in V(\{\sigma(1), \ldots, \sigma(k)\}), \\ & \forall_{i \in \{1,\ldots,k-1\}} : x_{\sigma(i)} = M^{\sigma}_{\sigma(i)}(V)\} \end{aligned}$$

for all $k \in \{2, \ldots, n\}$. We use the assumption of monotonicity to ensure that the sets over which the maximums are taken are nonempty. By construction, $M^{\sigma}(V) \in WPar(N)$. If a game is individually superadditive, then all marginal vectors belong to $IR(N)$.

Chapter 3

Convexity

3.1 Introduction

The notion of convexity for cooperative games with transferable utility was introduced by Shapley (1971) and is one of the most analysed properties in cooperative game theory. Many economic and combinatorial situations give rise to convex (or concave) cooperative games, such as airport games (cf. Littlechild and Owen (1973)), bankruptcy games (cf. Aumann and Maschler (1985)) and sequencing games (cf. Curiel et al. (1989)).

Convexity for TU games can be defined in a number of equivalent ways. One of these is by means of the *supermodularity property*, which has its origins outside the field of game theory. Vilkov (1977) and Sharkey (1981) have extended this property towards cooperative games with nontransferable utility to define *ordinal* and *cardinal convexity*, respectively. The supermodular interpretation of convexity also plays an important role in the context of effectivity functions (cf. Abdou and Keiding (1991)).

Economically more appealing than the supermodular interpretation of convexity are the definitions of convexity that are based on the concept of *marginal contributions*. In cooperative games with stochastic payoffs, this marginalistic interpretation of convexity has already been successfully applied (cf. Timmer et al. (2000) and Suijs (2000)). In this chapter, which is mainly based on Hendrickx et al. (2002), we build on the work originated by Ichiishi (1993) and consider three types of convexity for NTU games, which are based on three corresponding marginalistic convexity properties for TU games.

Although all five convexity properties for NTU games coincide within the subclass of TU games, they are not equivalent on the whole class of NTU games. In this chapter we analyse the relations between these convexity concepts.

This chapter is organised as follows. In section 2, we define the three marginalistic types of convexity for NTU games. In section 3, we investigate how the various types of convexity are related in general. In section 4, we analyse the relations between the convexity types in three-player games, while in section 5 we do this for some special classes of NTU games. Finally, in section 6, we study the relation between the various types of convexity and some rules.

3.2 Convexity

A TU game $v \in TU^N$ is called *convex* if it satisfies the following four equivalent conditions (cf. Shapley (1971) and Ichiishi (1981)):

$$\forall_{S,T\subset N} : v(S) + v(T) \leq v(S \cap T) + v(S \cup T), \tag{3.1}$$

$$\forall_{U\subset N}\forall_{S\subset T\subset N\setminus U} : v(S \cup U) - v(S) \leq v(T \cup U) - v(T), \tag{3.2}$$

$$\forall_{i\in N}\forall_{S\subset T\subset N\setminus\{i\}} : v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T), \tag{3.3}$$

$$\forall_{\sigma\in\Pi(N)} : m^{\sigma}(v) \in C(v). \tag{3.4}$$

Condition (3.1), which is called the *supermodularity property*, was originally stated in Shapley (1971) as the definition of convexity for TU games. Subsequently, Vilkov (1977) and Sharkey (1981) generalised this property to ordinal and cardinal convexity for NTU games, respectively. A game $V \in NTU^N$ is called *ordinally convex* if for all $S, T \subset N$ such that $S \neq \emptyset, T \neq \emptyset$ and for all $x \in \mathbb{R}^N$ such that $x_S \in V(S)$ and $x_T \in V(T)$ we have

$$x_{S\cap T} \in V(S \cap T) \text{ or } x_{S\cup T} \in V(S \cup T). \tag{3.5}$$

A game is called *cardinally convex* if for all coalitions $S, T \subset N$ such that $S \neq \emptyset, T \neq \emptyset$ we have[1]

$$V^{\circ}(S) + V^{\circ}(T) \subset V^{\circ}(S \cap T) + V^{\circ}(S \cup T), \tag{3.6}$$

where $V^{\circ}(S) = V(S) \times \{0^{N\setminus S}\}$ for all $S \subset N, S \neq \emptyset$ and $V^{\circ}(\emptyset) = \{0^N\}$.

[1]Cardinal convexity is only defined for $V \in NTU^N$ for which $V(N)$ is a convex set. Throughout this chapter, this condition is implicitly assumed when dealing with cardinal convexity. All results relating to the other types of convexity hold without this requirement.

In contrast to these *supermodular* definitions of convexity by Vilkov (1977) and Sharkey (1981), Ichiishi (1993) considers the *marginalistic* interpretation of convexity. We analyse three types of convexity for NTU games, based on the marginalistic properties (3.2)-(3.4).

First of all, we have *coalition merge convexity*[2], which generalises property (3.2). For $U = \emptyset$ and $S = T$, (3.2) is trivial and these cases can therefore be ignored when defining an analogous property for NTU games. If $S = \emptyset$, (3.2) is equivalent to superadditivity. Because we do not define $V(\emptyset)$ for NTU games, we require superadditivity as a separate condition. For $S \neq \emptyset$, (3.2) states that for any coalition U, the marginal contribution to the larger coalition T is larger than the marginal contribution to the smaller coalition S. In terms of allocations, this can be interpreted as follows: given the situation in which coalitions S and T have agreed upon an individually rational (and weak Pareto efficient) allocation of $v(S)$ and $v(T)$ (say, p and q, respectively), if coalition U joins the smaller coalition S, then for any allocation r of $v(S \cup U)$ such that the players in S get at least their previous amount ($r_S \geq p$), it is possible for U to join the larger coalition T using allocation s of $v(T \cup U)$, which gives the players in T at least their previous amount ($s_T \geq q$) and makes all players in U better off than in case they join S ($s_U \geq r_U$). Using this interpretation of (3.2), we can now define an analogous property for NTU games.

A game $V \in NTU^N$ is called *coalition merge convex*, if it is superadditive and it satisfies the *coalition merge property*, ie, for all coalitions $U \subset N$ such that $U \neq \emptyset$ and all $S \subsetneqq T \subset N \backslash U$ such that $S \neq \emptyset$ the following statement is true: for all $p \in WPar(S) \cap IR(S)$, all $q \in V(T)$ and all $r \in V(S \cup U)$ such that $r_S \geq p$, there exists an $s \in V(T \cup U)$ such that

$$\begin{cases} s_i \geq q_i & \text{for all } i \in T, \\ s_i \geq r_i & \text{for all } i \in U. \end{cases} \tag{3.7}$$

As a result of comprehensiveness, it makes no differences whether we require the coalition merge property for all $q \in V(T)$ or only for $q \in WPar(T) \cap IR(T)$.

The extension of (3.3) towards NTU games goes in a similar manner: a game $V \in NTU^N$ is called *individual merge convex* if it is individually superadditive and it satisfies the *individual merge property*, ie, for all $k \in N$ and all $S \subsetneqq T \subset N \backslash \{k\}$ such that $S \neq \emptyset$, the following statement is true: for all $p \in WPar(S) \cap IR(S)$, all

[2]This notion is introduced for stochastic cooperative games in Suijs and Borm (1999). The name *coalition merge convexity* and the subsequent names *individual merge convexity* and *marginal convexity* are from Timmer et al. (2000).

$q \in V(T)$ and all $r \in V(S \cup \{k\})$ such that $r_S \geq p$ there exists an $s \in V(T \cup \{k\})$ such that

$$\begin{cases} s_i \geq q_i & \text{for all } i \in T, \\ s_k \geq r_k. \end{cases} \tag{3.8}$$

Finally, a game $V \in NTU^N$ is called *marginal convex* if for all $\sigma \in \Pi(N)$ we have

$$M^\sigma(V) \in C(V). \tag{3.9}$$

One important aspect of the five convexity properties defined in this section is that within the class of NTU games that correspond to TU games, they are all equivalent and coincide with TU convexity. Another aspect of these properties is that if a game satisfies some particular form of convexity, then all its subgames do.

3.3 Relations between the convexity notions

In this section we investigate the relations between the five types of convexity for NTU games that were presented in the previous section. For 2-player NTU games, all five types are equivalent to (individual) superadditivity. For general n-player NTU games, equivalence between the five types of convexity does not hold. The remainder of this section shows which relations do exist between these properties.

It follows immediately from the definitions that coalition merge convexity implies individual merge convexity. The following example shows that the reverse need not be the case.

Example 3.3.1 Consider the following NTU game with player set $N = \{1, 2, 3, 4\}$:

$$V(\{i\}) = (-\infty, 0] \text{ for all } i \in N,$$

$$V(S) = \{x \in \mathbb{R}^S \mid \max_{i \in S} x_i \leq 1\} \text{ if } S = \{1,2\} \text{ or } S = \{3,4\},$$

$$V(S) = \{x \in \mathbb{R}^S \mid \max_{i \in S} x_i \leq 0\} \text{ for other } S \subset N, |S| = 2,$$

$$V(\{1,2,3\}) = \{x \in \mathbb{R}^{\{1,2,3\}} \mid x_1 \leq 1, x_2 \leq 1, x_3 \leq 0\},$$

$$V(\{1,2,4\}) = \{x \in \mathbb{R}^{\{1,2,4\}} \mid x_1 \leq 1, x_2 \leq 1, x_4 \leq 0\},$$

$$V(\{1,3,4\}) = \{x \in \mathbb{R}^{\{1,3,4\}} \mid x_1 \leq 0, x_3 \leq 1, x_4 \leq 1\},$$

$$V(\{2,3,4\}) = \{x \in \mathbb{R}^{\{2,3,4\}} \mid x_2 \leq 0, x_3 \leq 1, x_4 \leq 1\},$$

$$V(N) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i \leq 3\}.$$

This game is not superadditive and therefore not coalition merge convex[3]: take $S = \{1,2\}, T = \{3,4\}$, then $(1,1) \in V(S)$ and $(1,1) \in V(T)$, but $(1,1,1,1) \notin V(S \cup T)$. This game does, however, satisfy individual merge convexity. First, individual superadditivity can easily be checked to be satisfied. Next, let $k \in N$, let $S \subsetneq T \subset N\backslash\{k\}$ be such that $S \neq \emptyset$ and let $p \in WPar(S) \cap IR(S), q \in V(T)$ and $r \in V(S \cup \{k\})$ be such that $r_S \geq p$. Define $s = (q, r_k) \in \mathbb{R}^{T \cup \{k\}}$. If $|T| = 3$, then we have $T \cup \{k\} = N$. Because $\sum_{i \in T} q_i \leq 2$ and $r_k \leq 1$ (which follows from $|S| \leq 2$), we have $\sum_{i \in N} s_i \leq 3$ and hence, $s \in V(N)$. If $T = \{1,2\}$ or $T = \{3,4\}$, then we have $|S| = 1$ and $r_k \leq 0$ and because of individual superadditivity, $s \in V(T \cup \{k\})$. Finally, for other coalitions T with $|T| = 2$, we have $\max_{i \in T} q_i \leq 0$, $r_k \leq 1$ and therefore $s \in V(T \cup \{k\})$. Hence, V satisfies the individual merge property. $\triangleleft$

The following theorem shows that individual merge convexity implies marginal convexity.

Theorem 3.3.1 *Let $V \in NTU^N$. If V is individual merge convex, then it is marginal convex.*

Proof: Assume that V is individual merge convex and let $\sigma \in \Pi(N)$. To simplify notation, assume without loss of generality that $\sigma(i) = i$ for all $i \in N$. We prove that $M^\sigma(V) \in C(V)$ by induction on the player set. For this, we define for $k \in \{1, \ldots, n\}$ the subgame (N^k, V^k) where $N^k = \{1, \ldots, k\}$ and $V^k(S) = V(S)$ for all $S \subset N^k, S \neq \emptyset$. $M^{\sigma,k}(V^k)$ denotes the marginal vector in (N^k, V^k) that corresponds to the ordering σ restricted to the first k positions. For $k = 1$, $M^{\sigma,k}(V^k) \in C(V^k)$ by construction. Next, let $k \in \{2, \ldots, n\}$ and assume that $M^{\sigma,k-1}(V^{k-1}) \in C(V^{k-1})$. We show that $M^{\sigma,k}(V^k) \in C(V^k)$, ie, no coalition has an incentive to leave the "grand" coalition N^k. Define $T = \{1, \ldots, k-1\}$ and let $S \subsetneq T, S \neq \emptyset$. Then it suffices to show that coalitions S, T, $\{k\}$, $T \cup \{k\}$ and $S \cup \{k\}$ have no incentive to split off:

[3]One can even construct an individual merge convex game that is superadditive, but which does not satisfy the coalition merge property.

- Because $M^{\sigma,k-1}(V^{k-1}) \in C(V^{k-1})$, by definition there does not exist a $y \in V(S)$ such that $y > M_S^{\sigma,k-1}(V^{k-1})$. By construction, $M_S^{\sigma,k}(V^k) = M_S^{\sigma,k-1}(V^{k-1})$, so there does not exist a $y \in V(S)$ such that $y > M_S^{\sigma,k}(V^k)$. Hence, coalition S has no incentive to leave N^k when the payoff vector is $M^{\sigma,k}(V^k)$. The same argument holds for coalition T.

- Player k will not deviate on his own, because individual merge convexity implies individual superadditivity and hence, $M^{\sigma,k}(V^k) \in IR(V^k)$.

- Because $M^{\sigma,k}(V^k) \in WPar(N^k)$, there exists no $y \in V^k(N^k)$ such that $y > M^{\sigma,k}(V^k)$ and hence, the "grand" coalition $T \cup \{k\}$ has no incentive to deviate.

- Finally, we show that coalition $S \cup \{k\}$ has no incentive to split off. Define $R = \{r \in V(S \cup \{k\}) \mid r_S \geq M_S^{\sigma,k}(V^k)\}$ to be the set of allocations in $V(S \cup \{k\})$ according to which the players in S get at least the amount they get according to the marginal vector $M^{\sigma,k}(V^k)$. If $R = \emptyset$, then $S \cup \{k\}$ will be satisfied with the allocation $M^{\sigma,k}(V^k)$. Because $M^{\sigma,k}(V^k) \in IR(N^k)$, it follows from the basic assumptions of an NTU game that R is closed and bounded, so if $R \neq \emptyset$, we can compute $\max\{r_k \mid r = (r_S, r_k) \in R\}$. Let $r \in R$ be a point in which this maximum is reached. Because $M^{\sigma,k-1}(V^{k-1}) \in C(V^{k-1})$, we must have $M_S^{\sigma,k}(V^k) \notin V(S)$ or $M_S^{\sigma,k}(V^k) \in WPar(S)$. Let p be the intersection point of the line segment between 0 and $M_S^{\sigma,k}(V^k)$ and the set $WPar(S) \cap IR(S)$. By construction, $r \in V(S \cup \{k\})$ is such that $r_S \geq p$.
  Next, take $q = M^{\sigma,k-1}(V^{k-1}) \in V(T)$. As a result of individual merge convexity and comprehensiveness, there exists an $s \in V(T \cup \{k\})$ such that $s_T = q$ and $s_k \geq r_k$. Because $s_T = M^{\sigma,k-1}(V^{k-1})$, it follows from the construction of $M^{\sigma,k}(V^k)$ that $M_k^{\sigma,k}(V^k) \geq s_k$. But then, $M_k^{\sigma,k}(V^k) \geq r_k$. We constructed r_k as the maximum amount player k can obtain by cooperating with coalition S, while giving each player $i \in S$ at least $M_i^{\sigma,k}(V^k)$. Hence, we conclude that there does not exist a $y \in V(S \cup \{k\})$ such that $y_i > M_i^{\sigma,k}(V^k)$ for all $i \in S \cup \{k\}$.

From these four cases we conclude that $M^{\sigma,k}(V^k) \in C(V^k)$ and by induction on k, we obtain $M^\sigma(V) \in C(V)$. $\square$

In Example 3.3.2 we show that the reverse implication of Theorem 3.3.1 need not hold.

Example 3.3.2 The following game with player set $N = \{1, 2, 3\}$ is the NTU analogue of Example 4.6 in Timmer et al. (2000), which is a cooperative game with stochastic payoffs:

$$V(\{i\}) = (-\infty, 0] \text{ for all } i \in N,$$

$$V(\{1,2\}) = \{x \in \mathbb{R}^{\{1,2\}} \,|\, x_1 + x_2 \leq 3\},$$

$$V(\{1,3\}) = \{x \in \mathbb{R}^{\{1,3\}} \,|\, x_1 + x_3 \leq 2\},$$

$$V(\{2,3\}) = \{x \in \mathbb{R}^{\{2,3\}} \,|\, x_2 + x_3 \leq 6\},$$

$$V(N) = \{x \in \mathbb{R}^N \,|\, \frac{x_1}{6} + \frac{x_2}{10} + \frac{x_3}{14} \leq 1\}.$$

The marginal vectors of this games are stated in the following table.

σ	$(1,2,3)$	$(1,3,2)$	$(2,1,3)$	$(2,3,1)$	$(3,1,2)$	$(3,2,1)$
$M^\sigma(V)$	$(0,3,\frac{49}{5})$	$(0,\frac{60}{7},2)$	$(3,0,7)$	$(\frac{24}{7},0,6)$	$(2,\frac{20}{3},0)$	$(\frac{12}{5},6,0)$

The core is given by

$$C(V) = \{x \in \mathbb{R}^N_+ \,|\, \frac{x_1}{6} + \frac{x_2}{10} + \frac{x_3}{14} = 1, x_1 + x_3 \geq 3, x_1 + x_3 \geq 2, x_2 + x_3 \geq 6\}.$$

It is easy to check that $M^\sigma(V) \in C(V)$ for all $\sigma \in \Pi(N)$ and hence, V is marginal convex. Next, we show that this game is not individual merge convex. Take $k = 1, S = \{2\}, T = \{2,3\}$ and take $p = 0 \in WPar(S) \cap IR(S), q = (6,0) \in V(T)$ and $r = (3,0) \in V(S \cup \{k\})$. Note that $r_S \geq p$. Suppose V is individual merge convex. Then there exists an $s \in V(T \cup \{k\})$ such that (3.8) holds, ie, $s_2 \geq 6, s_3 \geq 0$ and $s_1 \geq 3$. But $s \in V(T \cup \{k\})$ implies $\frac{s_1}{6} + \frac{s_2}{10} + \frac{s_3}{14} \leq 1$, which gives a contradiction. Hence, V is not individual merge convex. $\triangleleft$

In the following example, we prove that ordinal convexity does not imply any of the other four types of convexity. This example disproves Theorem 2.2.3 in Ichiishi (1993), which states that in an ordinally convex NTU game, all marginal vectors are in the core.

Example 3.3.3 Consider the following NTU game with player set $N = \{1, 2, 3\}$:

$$V(\{i\}) = (-\infty, 0] \text{ for all } i \in N,$$

$$V(\{1,2\}) = \{x \in \mathbb{R}^{\{1,2\}} \,|\, x_1 \leq 0, x_2 \leq 2\},$$

$$V(\{1,3\}) = \{x \in \mathbb{R}^{\{1,3\}} \,|\, x_1 + x_3 \leq 1\},$$

$$V(\{2,3\}) = \{x \in \mathbb{R}^{\{2,3\}} \,|\, x_2 \leq 0, x_3 \leq 0\},$$

$$V(N) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i \leq 2\}.$$

This game V is ordinally convex: let $S, T \subset N$ be such that $S \neq \emptyset, T \neq \emptyset$ and let $x \in \mathbb{R}^N$ be such that $x_S \in V(S)$ and $x_T \in V(T)$. We distinguish between four cases. If $S \subset T$ or $T \subset S$, then (3.5) is trivially satisfied. If $S \cap T = \emptyset$, then (3.5) is equivalent to superadditivity, which is satisfied by this game. If $S = \{1,2\}$ and $T = \{1,3\}$, then $x_1 \leq 0$ and hence, $x_{S \cap T} \in V(S \cap T)$. Otherwise, $\sum_{i \in N} x_i \leq 2$ and hence, $x_{S \cup T} \in V(S \cup T)$. From these four cases we conclude that (3.5) is satisfied and V is ordinally convex.

However, this game is not marginal convex, because the marginal vector corresponding to $\sigma = (1,2,3), M^\sigma(V) = (0,2,0)$, does not belong to the core, because coalition $\{1,3\}$ has an incentive to leave the grand coalition. Using Theorem 3.3.1, we conclude that V is neither coalition merge nor individual merge convex. Furthermore, this game is not cardinally convex: $(0,2,0) \in V^\circ(\{1,2\})$ and $(0,0,1) \in V^\circ(\{1,3\})$, but $(0,2,0) + (0,0,1) = (0,2,1) \notin V^\circ(\{1\}) + V^\circ(N)$. $\triangleleft$

Next, we show that ordinal convexity is not implied by any of the other four types of convexity.

Example 3.3.4 Consider the following NTU game with player set $N = \{1,2,3,4\}$:

$$V(\{i\}) = (-\infty, 0] \text{ for all } i \in N,$$

$$V(S) = \{x \in \mathbb{R}^S \mid \max_{i \in S} x_i \leq 1\} \text{ for all } S \subset N, |S| = 2,$$

$$V(S) = \{x \in \mathbb{R}^S \mid \sum_{i \in S} x_i \leq 4\} \text{ for all } S \subset N, |S| = 3,$$

$$V(N) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i \leq 7\}.$$

First, we show that this game is not ordinally convex. Consider $S = \{1,2,3\}, T = \{2,3,4\}$ and $x = (4,-3,3,4) \in \mathbb{R}^N$. Then we have both $x_S \in V(S)$ and $x_T \in V(T)$, but neither $x_{S\cap T} \in V(S\cap T)$ nor $x_{S\cup T} \in V(S\cup T)$. Hence, (3.5) is not satisfied and V is not ordinally convex.

Next, we show that V is coalition merge convex. Let $U \subset N, U \neq \emptyset$ and let $S \subsetneqq T \subset N\backslash U$ be such that $S \neq \emptyset$. Let $p \in WPar(S)\cap IR(S)$, let $q \in V(T)$ and let $r \in V(S\cup U)$ be such that $r_S \geq p$. Define $s = (q, r_U)$. If $|T| = 3$ and $|U| = 1$, then $\sum_{i\in T} q_i \leq 4$ and $r_U \leq 3$. If $|T| = 2$ and $|U| = 2$, then $\sum_{i\in T} q_i \leq 2$ and $\sum_{i\in U} r_i \leq 4$. In both cases, we have $\sum_{i\in T\cup U} s_i \leq 7$ and hence, $s \in V(T\cup U) = V(N)$. In case $|T| = 2$ and $|U| = 1$, we have $\sum_{i\in T} q_i \leq 2$ and $r_U \leq 1$ and hence, $\sum_{i\in T\cup U} s_i \leq 3$, implying $s \in V(T\cup U)$. Noting that V is superadditive, we conclude that this game is coalition merge convex, and hence, also individual merge and marginal convex.

Finally, we show that V is cardinally convex. Let $S, T \subset N$ be such that $S \neq \emptyset, T \neq \emptyset$ and let $x^S \in V^\circ(S), x^T \in V^\circ(T)$. If $S \subset T$ or $T \subset S$, then (3.6) is trivially satisfied. If $S \cap T = \emptyset$, then (3.6) follows from superadditivity. We distinguish between three further cases. First, if $|S| = |T| = 3$, then $|S\cap T| = 2$ and $S\cup T = N$. Take $x^{S\cap T} = e^{S\cap T} \in V^\circ(S\cap T)$ and define $x = x^S + x^T - x^{S\cap T}$. Then $\sum_{i\in S\cup T} x_i = \sum_{i\in S} x_i^S + \sum_{i\in T} x_i^T - 2 \leq 4 + 4 - 2 = 6$. Hence, $x \in V^\circ(S\cup T)$. Second, if $|S| = 2, T = |3|$, then $|S\cap T| = 1$ and $S\cup T = N$. Take $x^{S\cap T} = 0^N \in V^\circ(S\cap T)$ and define x as before. Then $\sum_{i\in S\cup T} x_i \leq 2+4-0 = 6$ and hence, $x \in V^\circ(S\cup U)$. Third, if $|S| = |T| = 2$, then $|S\cap T| = 1$ and $|S\cup T| = 3$. Take $x^{S\cap T} = 0^N \in V^\circ(S\cap T)$ and define x as before. Then $\sum_{i\in S\cup T} x_i \leq 2+2-0 = 4$ and hence, $x \in V^\circ(S\cup U)$. From these three cases we conclude that V is cardinally convex. $\triangleleft$

From the previous two examples we conclude that ordinal convexity is independent of the other four types of convexity. The example below shows that cardinal convexity does not imply any of the marginalistic types of convexity.

Example 3.3.5 Consider the following NTU game with player set $N = \{1, 2, 3, 4\}$:

$$V(\{i\}) = (-\infty, 0] \text{ for all } i \in N,$$

$$V(\{1,2\}) = \{x \in \mathbb{R}^{\{1,2\}} \mid x_1 + x_2 \leq 2, x_2 \leq 1\},$$

$$V(S) = \{x \in \mathbb{R}^S \mid \max_{i\in S} x_i \leq 0\} \text{ for other } S \subset N, |S| = 2,$$

$$V(\{1,2,3\}) = \{x \in \mathbb{R}^{\{1,2,3\}} \mid x_1 + x_2 + x_3 \leq 2, x_3 \leq 2\},$$

$$V(\{1,2,4\}) = \{x \in \mathbb{R}^{\{1,2,4\}} \mid x_1 + x_2 + x_4 \leq 2, x_4 \leq 1\},$$

$$V(S) = \{x \in \mathbb{R}^S \mid \max_{i \in S} x_i \leq 0\} \text{ for other } S \subset N, |S| = 3,$$

$$V(N) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i \leq 2, x_3 \leq 2, x_4 \leq 1\}.$$

For the cardinal convexity property (3.6), only the case with $S = \{1,2,3\}$ and $T = \{1,2,4\}$ is nontrivial. Let $x^S \in V^\circ(S), x^T \in V^\circ(T)$. Because $(1,1,0,0) \in V^\circ(S \cap T)$, it suffices to show that $x = x^S + x^T - (1,1,0,0) \in V^\circ(S \cup T) = V(N)$. Now,

$$\sum_{i \in N} x_i = \sum_{i \in S} x_i^S + \sum_{i \in T} x_i^T - 2 \leq 2 + 2 - 2 = 2,$$

$$x_3 = x_3^S \leq 2,$$

$$x_4 = x_4^T \leq 1.$$

Hence, $x \in V(N)$ and V is cardinally convex. For $\sigma = (1,2,3,4)$ we have $M^\sigma = (0,1,1,0)$. The players of coalition $\{1,2,4\}$ have an incentive to deviate from this vector, because the allocation $(\frac{1}{3}, \frac{4}{3}, \frac{1}{3}) \in V(\{1,2,4\})$ gives them a strictly higher payoff. Hence, $M^\sigma(V) \notin C(V)$ and V is not marginal convex, and therefore neither coalition merge nor individual merge convex. $\triangleleft$

Finally, we show that the three marginalistic convexity properties do not imply cardinal convexity.

Example 3.3.6 Consider the following NTU game with player set $N = \{1,2,3\}$:

$$V(\{i\}) = (-\infty, 0] \text{ for all } i \in N,$$

$$V(S) = \{x \in \mathbb{R}^S \mid \max_{i \in S} x_i \leq 1\} \text{ for } S \subset N, |S| > 1.$$

This game is a 1-corner game (see section 3.5.2) and it follows from Proposition 3.5.4 that V is coalition merge convex (and hence, individual merge and marginal convex as well). This game is, however, not cardinally convex: take $S = \{1,2\}, T = \{2,3\}$ and take $(1,1,0) \in V^\circ(S), (0,1,1) \in V^\circ(T)$. Then $(1,1,0) + (0,1,1) = (1,2,1) \notin V^\circ(S \cap T) + V^\circ(S \cup T)$. $\triangleleft$

Summarising all the results in this section, the five types of convexity for NTU games are related as is depicted in Figure 3.1. Cardinal convexity is abbreviated to card-convex, coalition merge convexity to cm-convex, individual merge convexity to im-convex, ordinal convexity to ord-convex and marginal convexity to m-convex. An arrow from one type of convexity to another indicates that the former implies the latter. Where an arrow is absent, such an implication does not hold in general.

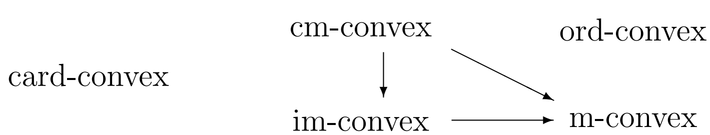


Figure 3.1: Relations between the convexity notions

3.4 Three-player games

The results in Figure 3.1 hold for general n-player NTU games. In this section, we consider the relations between the five types of convexity for 3-player NTU games. First, we prove that in 3-player NTU games, individual merge convexity implies coalition merge convexity.

Proposition 3.4.1 *Let $V \in NTU^N$ such that $|N| = 3$. If V is individual merge convex, then it is coalition merge convex.*

Proof: Assume that V is individual merge convex. Then V is individually superadditive, and because there are only three players, superadditive. For the coalition merge property, if $|U| = 1$, then (3.7) is equivalent to (3.8). For $|U| > 1$, we cannot find coalitions S and T such that $S \subsetneqq T \subset N \backslash U$ and $S \neq \emptyset$. Hence, the coalition merge property is satisfied. $\square$

Next, we show that in 3-player games, coalition merge convexity implies ordinal convexity.

Proposition 3.4.2 *Let $V \in NTU^N$ such that $|N| = 3$. If V is coalition merge convex, then it is ordinally convex.*

Proof: Assume that V is coalition merge convex. Let $S_1, S_2 \subset N$ be such that $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$. If $S_1 \subset S_2$ or $S_2 \subset S_1$, then (3.5) is trivially satisfied. If $S_1 \cap S_2 = \emptyset$, then (3.5) is satisfied because V is superadditive. Otherwise, let $x \in \mathbb{R}^N$ be such that $x_{S_1} \in V(S_1)$ and $x_{S_2} \in V(S_2)$ and suppose $x_{S_1 \cap S_2} \notin V(S_1 \cap S_2)$. Then $x_{S_1 \cap S_2} > 0$ because $|S_1 \cap S_2| = 1$. Next, define $U = S_2 \backslash S_1$, $S = S_1 \cap S_2$ and $T = S_1$ and take $p = 0 \in WPar(S) \cap IR(S)$, $q = x_{S_1} \in V(T)$ and $r = x_{S_2} \in V(S \cup U)$. Then $r_S = x_{S_1 \cap S_2} > 0 = p$. Because V is coalition merge convex, there exists an $s \in V(T \cup U) = V(N)$ such that $s \geq (q, r_U) = (x_T, x_U) = x_{S_1 \cup S_2}$. Hence, $x_{S_1 \cup S_2} \in V(N) = V(S_1 \cup S_2)$ and V is ordinally convex. □

The following example shows that in 3-player NTU games, marginal convexity need not imply ordinal convexity.

Example 3.4.1 Consider the following NTU game with player set $N = \{1, 2, 3\}$:

$$V(\{i\}) = (-\infty, 0] \text{ for all } i \in N,$$

$$V(S) = \{x \in \mathbb{R}^S \mid \max_{i \in S} x_i \leq 1\} \text{ for all } S \subset N, |S| = 2,$$

$$V(N) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i \leq 2\}.$$

The marginal vectors of this game are

σ	$(1,2,3)$	$(1,3,2)$	$(2,1,3)$	$(2,3,1)$	$(3,1,2)$	$(3,2,1)$
M^σ	$(0,1,1)$	$(0,1,1)$	$(1,0,1)$	$(1,0,1)$	$(1,1,0)$	$(1,1,0)$

and the core is

$$C(V) = \{(1,1,0), (1,0,1), (0,1,1)\}.$$

This game is marginal convex. For ordinal convexity, consider $S = \{1, 2\}, T = \{2, 3\}$ and $x = (1, 1, 1) \in \mathbb{R}^N$. Then we have both $x_S \in V(S)$ and $x_T \in V(T)$, but neither $x_{S \cap T} \in V(S \cap T)$ nor $x_{S \cup T} \in V(S \cup T)$. Hence, V is not ordinally convex. ◁

Finally, we show that in 3-player games, cardinal convexity is stronger than coalition merge convexity.

Proposition 3.4.3 *Let $V \in NTU^N$ be such that $|N| = 3$. If V is cardinally convex, then it is coalition merge convex.*

Proof: Assume that V is cardinally convex. Then it is superadditive. For the coalition merge property, let $U \subset N$ be such that $U \neq \emptyset$ and let $S \subsetneqq T \subset N \backslash U$ be such that $S \neq \emptyset$. Let $p \in WPar(S) \cap IR(S)$, $q \in V^{\circ}(T)$ and $r \in V^{\circ}(S \cup U)$ be such that $r_S \geq p$. Because $|S| = 1$, we have $p = 0$ and hence, $r_S \geq 0$. Next, define $\hat{S} = S \cup U$. Then $q + r \in V^{\circ}(\hat{S}) + V^{\circ}(T)$ and because of cardinal convexity, there exists an $s \in V^{\circ}(\hat{S} \cap T) + V^{\circ}(\hat{S} \cup T)$ such that $s \geq q + r$. Because $|\hat{S} \cap T| = |S| = 1$, $V^{\circ}(\hat{S} \cap T) = \mathbb{R}_{-} \times 0^{N \backslash (\hat{S} \cap T)}$ and hence, $s \in V^{\circ}(\hat{S} \cup T) = V(N) = V(T \cup U)$. Furthermore, $s_T = (s_S, s_{T \backslash S}) \geq (r_S + q_S, q_{T \backslash S}) \geq q$ and $s_U \geq r_U$. So s satisfies (3.7) and V is coalition merge convex. $\square$

As a corollary, we obtain that in 3-player NTU games, cardinal convexity implies individual merge, marginal and ordinal convexity as well.

Combining the results of this section with some results of the previous section, in Figure 3.2 we depict all the relations between the five types of convexity for 3-player games. To keep the picture clear, the arrows from cardinal convexity to ordinal and marginal convexity have been omitted.

$$\begin{tikzcd}
 & \text{cm-convex} \arrow[r] \arrow[dd, leftrightarrow] \arrow[ddr] & \text{ord-convex} \\
\text{card-convex} \arrow[ur] \arrow[dr] & & \\
 & \text{im-convex} \arrow[r] \arrow[uur] & \text{m-convex}
\end{tikzcd}$$

Figure 3.2: Relations between the convexity notions, three players

3.5 Special classes of games

In this section, we look at our convexity notions in some specific classes of NTU games.

3.5.1 Hyperplane games

A *hyperplane game* is an NTU game $V \in NTU^N$ such that for all coalitions $S \subset N, S \neq \emptyset$ we have

$$V(S) = \{x \in \mathbb{R}^S \,|\, x^{\top} a^S \leq b^S\} \tag{3.10}$$

for certain $a^S \in \overset{\circ}{\Delta}{}^S = \{x \in \mathbb{R}^S \mid \sum_{i \in S} x_i = 1, x > 0\}$ and $b^S \in \mathbb{R}$. Note that every entry of a^S must be positive to ensure boundedness of $V(S) \cap \mathbb{R}^S_+$. We denote the class of all hyperplane games with player set N by $\mathcal{H}^N$. A property of hyperplane games that we are going to use later on, is that these games possess a convex core.

Lemma 3.5.1 *Let $V \in \mathcal{H}^N$. Then $C(V)$ is a convex set.*

Proof: Let a^S, b^S for all $S \subset N, S \neq \emptyset$ be as in (3.10). Then

$$\begin{aligned} C(V) &= \{x \in V(N) \,|\, \forall_{S \subset N, S \neq \emptyset} \nexists_{y \in V(S)} : y > x_S\} \\ &= \bigcap_{S \subset N, S \neq \emptyset} \{x \in \mathbb{R}^N \,|\, \nexists_{y \in V(S)} : y > x_S\} \cap V(N) \\ &= \bigcap_{S \subsetneq N, S \neq \emptyset} \{x \in \mathbb{R}^N \,|\, x_S^\top a^S \geq b^S\} \cap \{x \in \mathbb{R}^N \,|\, x^\top a^N = b^N\}. \end{aligned}$$

$C(V)$ is the intersection of a finite number of convex sets and is hence convex. $\square$

A *parallel hyperplane game* is a hyperplane game $V \in \mathcal{H}^N$ such that the projection of a^N onto $\mathring{\Delta}^S$ equals a^S for all coalitions $S \subset N, S \neq \emptyset$. A parallel hyperplane game can be viewed as a TU game in which each player's utility is multiplied by a certain positive factor. We denote the class of parallel hyperplane games with player set N by $\mathcal{P}^N$.

The next lemma shows that parallel hyperplane games are the only hyperplane games that can be individually superadditive. As a result, hyperplane games that are not parallel cannot be coalition merge, individual merge, ordinal or cardinal convex.

Lemma 3.5.2 *Let $V \in \mathcal{H}^N$. If V is individually superadditive, then it belongs to $\mathcal{P}^N$.*

Proof: Assume that V is individually superadditive and for all $S \subset N, S \neq \emptyset$, let a^S, b^S be as in (3.10). Let $S \subset N, S \neq \emptyset$. Take $p \in V(S)$ and let $i, j \in S$. Construct for all $\alpha \in \mathbb{R}$ the vector $p_\alpha = p + \alpha(\frac{a_i^S}{a_j^S} e^j - e^i)$, where e^j and e^i are unit vectors in $\mathbb{R}^S$. Then

$$\begin{aligned} p_\alpha^\top a^S &= p^\top a^S + \alpha(\frac{a_i^S}{a_j^S}(e^j)^\top a^S - (e^i)^\top a^S) \\ &= p^\top a^S + \alpha(\frac{a_i^S}{a_j^S} a_j^S - a_i^S) \\ &= p^\top a^S \end{aligned}$$

$$\leq \ b^S$$

for all $\alpha \in \mathbb{R}$ and hence, $p_\alpha \in V(S)$. Next, define $q_\alpha = (p_\alpha, 0^{N\setminus S})$ for all $\alpha \in \mathbb{R}$. Applying individual superadditivity $|N\setminus S|$ times yields $q_\alpha \in V(N)$. Hence,

$$q_\alpha^\top a^N = p^\top a_S^N + \alpha\left(\frac{a_i^S}{a_j^S}(e^j)^\top a_S^N - (e^i)^\top a_S^N\right) \leq b^N$$

for all $\alpha \in \mathbb{R}$. The inequality can only hold for all $\alpha \in \mathbb{R}$ if the expression between parentheses equals zero. Therefore $\frac{a_i^S}{a_j^S} = \frac{a_i^N}{a_j^N}$. Hence, a^S is the projection of a^N onto $\mathring{\Delta}^S$ and $V \in \mathcal{P}^N$. $\square$

The following lemma relates the five convexity properties within the class of parallel hyperplane games.

Lemma 3.5.3 *Within $\mathcal{P}^N$, coalition merge, individual merge, marginal, ordinal and cardinal convexity coincide.*

Proof: First of all, note that all five convexity properties are scale invariant: if V satisfies some form of convexity, then so does V^w for every vector of scale factors $w \in \mathbb{R}^N_{++}$, where $V^w(S) = \{(w_i x_i)_{i\in S} \,|\, x \in V(S)\}$ for all $S \subset N, S \neq \emptyset$. In a parallel hyperplane game $V \in \mathcal{P}^N$, one can choose w in such a way that V^w corresponds to a TU game. From this the assertion follows. $\square$

The relations between the various forms of convexity for hyperplane games are summarised in Figure 3.3. For simplicity, the double arrow between cardinal and ordinal convexity and the arrow from cardinal to marginal convexity have been omitted. It follows from Lemmas 3.5.2 and 3.5.3 that within the class $\mathcal{H}^N$, coalition merge, individual merge, ordinal and cardinal convexity coincide. Because there are hyperplane games that are marginal convex, but not parallel, marginal convexity is weaker than the other four types of convexity.

$$\begin{tikzcd}
 & \text{cm-convex} \arrow[r, leftrightarrow] \arrow[dd, leftrightarrow] \arrow[ddr] & \text{ord-convex} \arrow[dd] \\
\text{card-convex} \arrow[ur, leftrightarrow] \arrow[dr, leftrightarrow] & & \\
 & \text{im-convex} \arrow[r] \arrow[uur, leftrightarrow] & \text{m-convex}
\end{tikzcd}$$

Figure 3.3: Relations between convexity notions, hyperplane games

3.5.2 1-corner games

An NTU game is called a *1-corner game* if $V(S) = \{x \in \mathbb{R}^S \mid x \leq u^S\}$ for some $u^S \in \mathbb{R}^S$ for all $S \subset N, S \neq \emptyset$. We denote the class of 1-corner games with player set N by $\mathcal{C}^N$. Monotonicity implies that for all $S \subset T \subset N, S \neq \emptyset$ we must have $u^T_S \geq u^S$. From this, superadditivity readily follows.

The core of a 1-corner game is given by (cf. Otten (1995)):

$$C(V) = \bigcup_{\sigma \in \Pi(N)} \{x \in V(N) \mid x \geq M^\sigma(V)\} \tag{3.11}$$

In the following proposition we show that all 1-corner games are coalition merge convex.

Proposition 3.5.4 *Let $V \in \mathcal{C}^N$. Then V is coalition merge convex.*

Proof: Let $U \subset N$ be such that $U \neq \emptyset$, let $S \subsetneqq T \subset N \backslash U$ be such that $S \neq \emptyset$ and let $p \in WPar(S) \cap IR(S), q \in V(T)$ and $r \in V(S \cup U)$ be such that $r_S \geq p$. Then it suffices to show that $(q, r_U) \in V(T \cup U)$. First, $q \in V(T)$, so $q \leq u^T$. Similarly, $r \leq u^{S \cup U}$ and hence, $r_U \leq u^{S \cup U}_U$. Because of monotonicity, we have $q \leq u^{T \cup U}_T$ and $r_U \leq u^{T \cup U}_U$. Therefore, $(q, r_U) \leq u^{T \cup U}$ and $(q, r_U) \in V(T \cup U)$. □

It can be shown in a similar fashion that every 1-corner game is ordinally convex. However, a 1-corner game need not be cardinally convex, as is illustrated by Example 3.3.6.

3.5.3 Bargaining games

A *bargaining situation* is a pair (F, d) where $F \subset \mathbb{R}^N$ is a closed, convex and comprehensive set of attainable utility vectors and $d \in F$ is a disagreement point such that there exists a $y \in F$ with $y > d$.

A bargaining situation with $d = 0$ gives rise to the *bargaining game* V with $V(S) = \mathbb{R}^S_-$ for all $S \subsetneqq N, S \neq \emptyset$ and $V(N) = F$. We denote the class of bargaining games with player set N by $\mathcal{B}^N$.

Proposition 3.5.5 *Let $V \in \mathcal{B}^N$. Then V satisfies all five convexity properties.*

Proof: Define the game $W \in \mathcal{C}^N$ by $W(S) = \mathbb{R}^S_-$ for all coalitions $S \subset N, S \neq \emptyset$. Then W trivially satisfies all five convexity properties. Because $V(S) = W(S)$ for all $S \subsetneqq N, S \neq \emptyset$ and $V(N) \supsetneqq W(N)$, it follows from the definitions (3.5)-(3.9) that V satisfies all five convexity properties as well. $\square$

3.6 Relations between convexity and some rules

In this section we investigate how some rules and set-valued solutions on subclasses of NTU^N relate to our convexity notions.

3.6.1 The MC value

The *marginal based compromise value* or *MC value* was introduced in Otten et al. (1998) and is defined by

$$MC(V) = \alpha_V \sum_{\sigma \in \Pi(N)} M^\sigma(V), \tag{3.12}$$

where $\alpha_V = \max\{\alpha \in \mathbb{R}_+ \mid \alpha \sum_{\sigma\in\Pi(N)} M^\sigma(V) \in V(N)\}$.

Proposition 3.6.1 *Let $V \in NTU^N$. If V is marginal convex and belongs to $\mathcal{H}^N$, $\mathcal{C}^N$ or $\mathcal{B}^N$, then $MC(V) \in C(V)$.*

Proof: Assume that V is marginal convex. For $V \in \mathcal{H}^N$ and $V \in \mathcal{C}^N$, the statement follows from Lemma 3.5.1 and equation (3.11), respectively. If $V \in \mathcal{B}^N$, then it is easily seen that the core includes the set on the right hand side of (3.11), from which $MC(V) \in C(V)$ follows. $\square$

3.6.2 The compromise value and semi-convexity

The *compromise value* for NTU games is introduced in Borm et al. (1992) and is an extension of the τ value for TU games (cf. Tijs (1981)). The compromise value is a compromise between two payoff vectors. The first one is the utopia vector $K(V)$, defined by

$$K_i(V) = \sup\{t \in \mathbb{R} \mid \exists_{a\in\mathbb{R}_+^{N\setminus\{i\}}} : (a,t) \in V(N), \nexists_{b\in V(N\setminus\{i\})} : b > a\}$$

for all $i \in N$. The second one is the minimal right vector $k(V)$, defined by

$$k_i(V) = \max_{S: i\in S} \rho_i^S(V)$$

for all $i \in N$, where $\rho_i^S(V)$ is the remainder for player i after giving the other members in S their utopia payoff:

$$\rho_i^S(V) = \sup\{t \in \mathbb{R} \,|\, \exists_{a\in\mathbb{R}^{S\setminus\{i\}}} : (t,a) \in V(S), a > K_{S\setminus\{i\}}(V)\}.$$

The following lemma comes from Borm et al. (1992).

Lemma 3.6.2 *Let $V \in NTU^N$ with $x \in C(V)$. Then $k(V) \leq x \leq K(V)$.*

A game $V \in NTU^N$ is called *compromise admissible* if $k(V) \leq K(V), k(V) \in V(N)$ and there does not exist a $b \in V(N)$ such that $b > K(V)$. In view of Lemma 3.6.2, every NTU game with a nonempty core is compromise admissible. For a compromise admissible game, the compromise value $T(V)$ is defined by

$$T(V) = \lambda_V K(V) + (1-\lambda_V)k(V),$$

where

$$\lambda_V = \max\{\lambda \in [0,1] \,|\, \lambda K(V) + (1-\lambda)k(V) \in V(N)\}.$$

A game $V \in NTU^N$ is called *semi-convex* if $k(V) = 0$.[4] For TU games, semi-convexity is implied by convexity and the next lemma states the corresponding result for NTU games.

Lemma 3.6.3 *Let $V \in NTU^N$. If V is marginal convex, then it is semi-convex.*

Proof: Assume that V is marginal convex. Let $i \in N$ and let $\sigma \in \Pi(N)$ be such that $\sigma(1) = i$. By construction, $M_i^\sigma(V) = 0$. Because of Lemma 3.6.2, we have $k_i(V) \leq M_i^\sigma(V) = 0$. On the other hand, $k_i(V) = \max_{S:i\in S} \rho_i^S(V) \geq \rho_i^{\{i\}}(V) = 0$. We conclude that $k_i(V) = 0$ for all $i \in N$ and V is semi-convex. □

As a corollary, we obtain the following proposition, in which compromise admissibility follows from nonemptiness of the core.

Proposition 3.6.4 *Let $V \in NTU^N$. If V is marginal convex, then it is compromise admissible and the compromise value is proportional to the utopia payoff vector.*

[4] Contrary to the TU case (cf. Driessen and Tijs (1985)), we do not require superadditivity in the definition of semi-convexity.

3.6.3 The bargaining set

The *bargaining set* for a game $V \in NTU^N$ is defined as (cf. Aumann and Maschler (1964))

$$\begin{aligned}\mathcal{M}(V) \;&= \{x \in I(V) \mid \;\; \forall_{i,j\in N}\forall_{S\subset N, i\in S, j\notin S}\forall_{y\in WPar(S), y>x_S} \\ &\qquad\qquad\qquad\quad \exists_{T\subset N, i\notin T, j\in T}\exists_{z\in WPar(T)} : z \geq (y_{S\cap T}, x_{T\setminus S})\}.\end{aligned}$$

The bargaining set consists of those imputations x such that whenever player i raises an objection against player j by cooperating with coalition S and promising the members of S more than they get according to x, player j can counter this objection by cooperating with coalition T, giving each player in $S \cap T$ at least the amount they are promised by i.

It is a well-known result that in TU games, this set is always nonempty and contains the core. For convex TU games, the bargaining set coincides with the core (cf. Maschler et al. (1972)). In NTU games, the bargaining set still contains the core, but there are games in which $\mathcal{M}(V)$ is empty. In the next example we show that even a strong form of convexity does not ensure $\mathcal{M}(V) = C(V)$.

Example 3.6.1 Consider the same game as in Example 3.3.6, which is coalition merge convex. The imputation $x = (\frac{1}{2}, \frac{1}{2}, 1)$ does not belong to the core, but we show that $x \in \mathcal{M}(V)$. By symmetry, we only have to look at objections of player 1 against player 3. Player 1 cannot object on his own, but only through coalition $S = \{1, 2\}$. The maximum payoff vector player 1 can promise is $y = (1, 1)$. But player 3 can counter this objection through coalition $T = \{2, 3\}$ and payoff vector $z = (1, 1)$. Hence, $x \in \mathcal{M}(V)$ although $x \notin C(V)$ and V is coalition merge convex. $\triangleleft$

Of course, there might be some subclass of NTU^N for which coalition merge convexity (or even a weaker form of convexity) implies $\mathcal{M}(V) = C(V)$. The proof in Solymosi (1999) for the corresponding TU result uses excess games and it might be interesting to investigate how this result can be extended to NTU games, and in particular, what definition of excess games can be used in this context.

Chapter 4

Monotonicity

4.1 Introduction

In cooperative game theory, many monotonicity properties have been introduced to analyse and characterise various rules. Roughly, a rule on a class of games or economic situations is monotonic if a certain change in some of the parameters (eg, player set or payoff function) completely determines the *direction* of change in a player's payoff.

For example, a rule $f : TU^N \rightarrow \mathbb{R}^N$ is called *strongly monotonic* (cf. Young (1985)) if for all $u, v \in TU^N$ and $i \in N$ such that $v(S) - v(S \backslash \{i\}) \leq u(S) - u(S \backslash \{i\})$ for all $S \subset N, i \in S$, we have $f_i(u) \geq f_i(v)$. So, if we have a game $v \in TU^N$ and take a second game $u \in TU^N$ in which all of player i's marginal contributions are higher, then according to f, player i should get a higher payoff. Young (1985) uses this monotonicity property, together with efficiency, to characterise the Shapley value.

An overview of various types of monotonicity can be found in Levinský (2000).

In this chapter, we consider two types of monotonicity. First, we generalise a well-known TU result on population monotonic allocation schemes to NTU games. The bulk of this chapter deals with sequencing situations, for which we introduce the concept of drop out monotonicity.

The concept of population monotonic allocation scheme (pmas) was introduced in Sprumont (1990). A pmas of a game is a scheme consisting of a set of allocations, one for each subgame. These allocations are all core elements of their corresponding subgames, and moreover, each player's payoff increases as the coalition to which he belongs increases in size. As a result of this monotonicity property, the allocation for the grand coalition can be seen as fair, since each coordinate is bounded from below by core elements of the subgames.

Sprumont (1990) shows that for convex TU games, every extended marginal vector is a pmas and poses the question, whether this result can be extended to NTU games. In section 2 of this chapter, which is based on Hendrickx (2003), we use the extensions of TU convexity that we presented in the previous chapter to answer this question. We show that in an individual merge convex game, each extended marginal vector is indeed a pmas, while marginal convexity is not sufficient.

In the remainder of this chapter, which is based on Fernández et al. (2002), we introduce and analyse the concept of drop out monotonicity in the context of sequencing situations. We say that a rule on a certain class of situations is *drop out monotonic* if applying the rule to a reduced situation, in which one of the players has left, yields an allocation, which, depending on the context, either makes all remaining players better off or all of them worse off than in the original situation.

We link this concept of drop out monotonicity to stability. A rule is called *stable* if it always generates a core element of the corresponding TU game.

If the games corresponding to the reduced situations are subgames of the original game, then a stable and drop out monotonic rule generates a pmas for the original game (cf. Sprumont (1990)). In the cases of linear production situations (cf. Owen (1975)), airport situations (cf. Littlechild and Owen (1973)) and holding situations (cf. Tijs et al. (2000)), the game corresponding to a reduced situation after one player drops out is a subgame of the original game. So here, the existence of stable and monotonic rules boils down to the existence of a pmas. Such pmas-es do not always exist for linear production games. However, for airport situations, the Shapley value induces one of many stable and drop out monotonic rules. For holding games, the rule which gives all gains to the so-called holding house keeper is a pmas.

The property of drop out monotonicity introduced here is inspired by the *fairness condition* introduced in Ambec and Sprumont (2002). They study the problem of water management from a game theoretical point of view: given a river of certain capacity flowing through a number of countries with certain demand for water, how should the water of the river be allocated?

The fairness condition states that whenever one of the countries ceases to demand water (drops out), all other countries should be better off. Contrary to the examples mentioned before, the reduced situation after a player drops out does not give rise to a subgame of the original game. Ambec and Sprumont show that there is a unique allocation rule which satisfies both stability (ie, generates a core element) and the fairness condition. This rule (the μ rule) is the marginal vector corresponding to the ordering of the countries along the river (from upstream to downstream).

As stated before, we study the drop out monotonicity property in the context of sequencing situations, as introduced in Curiel et al. (1989), in which there is also a natural ordering of the players. Indeed, in the most basic class of sequencing situations (with linear cost functions), a result similar to Ambec and Sprumont is established. Within a more general class of sequencing situations (with regular cost functions), it turns out that there is *at most* one stable and drop out monotonic rule, which must be the (analogue of the) μ rule. Finally, we introduce a class of sequencing situations with linear cost functions, in which one of the players faces a due date. It turns out that in this class, the μ rule is indeed stable and drop out monotonic if the processing times of the agents are equal.

This chapter is organised as follows. In section 2, we consider pmas-es for NTU games and show that individual merge convexity is sufficient to ensure that each extended marginal vector is a pmas. In section 3, we introduce the basic sequencing model and define the μ rule. In section 4, we define drop out monotonicity for sequencing situations and show that if the cost functions are regular, there can be at most one stable and drop out monotonic rule. In section 5, we show that in the class of sequencing situations with linear cost functions in which one of the players faces a due date, the μ rule is stable and drop out monotonic.

4.2 Population monotonic allocation schemes

A *population monotonic allocation scheme* or *pmas* for a TU game $v \in TU^N$ is a collection of vectors $(y^S)_{S\subset N, S\neq\emptyset}$, where for all $S \subset N, S \neq \emptyset$ we have $y^S \in \mathbb{R}^S$ such that

$$\sum_{i\in S} y_i^S = v(S) \tag{4.1}$$

and

$$y_i^S \leq y_i^T \tag{4.2}$$

for all $\emptyset \neq S \subset T \subset N$ and all $i \in S$. A pmas for an NTU game $V \in NTU^N$ is a collection of vectors $(y^S)_{S\subset N, S\neq\emptyset}$ satisfying monotonicity condition (4.2) and the following efficiency condition, which generalises (4.1):

$$y^S \in WPar(S) \tag{4.3}$$

for all $S \subset N, S \neq \emptyset$.

As is the case for TU games, also in an NTU game a pmas induces a core element in every subgame, as is shown in the following lemma.

Lemma 4.2.1 *Let $V \in NTU^N$ and let $(y^S)_{S \subset N, S \neq \emptyset}$ be a pmas for V. Then $y^S \in C(V^S)$ for all $S \subset N, S \neq \emptyset$.*

Proof: Let $S \subset N, S \neq \emptyset$. Then by definition, $y^S \in V^S(S)$. Suppose that there exists a coalition $T \subset S, T \neq \emptyset$ and an allocation $x \in V^S(T)$ such that $x > y^S_T$. Then by (4.2), $x > y^T$, which contradicts $y^T \in WPar(T)$. Hence, $y^S \in C(V^S)$ □

The *extended marginal vector* of V with respect to ordering $\sigma \in \Pi(N)$ is the collection of vectors $(M^{\sigma_{|S}}(V^S))_{S \subset N, S \neq \emptyset}$, where $\sigma_{|S} \in \Pi(S)$ is such that $\sigma^{-1}(i) < \sigma^{-1}(j)$ implies $\sigma_{|S}^{-1}(i) < \sigma_{|S}^{-1}(j)$ for all $i, j \in S$.

Sprumont (1990) shows that for TU games, convexity implies that each of the extended marginal vectors consitutes a pmas. In his concluding paragraph, Sprumont poses the question whether this result can be generalised to NTU games. Moulin (1989) shows that in an ordinally convex NTU game, the extended marginal vector need not constitute a pmas. In the following example, we show that marginal convexity is not sufficient either.

Example 4.2.1 Consider the following NTU game with player set $N = \{1, 2, 3\}$:

$$V(\{i\}) = (-\infty, 0] \text{ for all } i \in N,$$
$$V(\{1,2\}) = \{x \in \mathbb{R}^{\{1,2\}} \mid 10x_1 + x_2 \leq 10\},$$
$$V(\{1,3\}) = \{x \in \mathbb{R}^{\{1,3\}} \mid x_1 + 10x_3 \leq 10\},$$
$$V(\{2,3\}) = \{x \in \mathbb{R}^{\{2,3\}} \mid x_2 + x_3 \leq 1\},$$
$$V(N) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i \leq 11\}.$$

It is readily verified that this game is marginal convex. Take $\sigma = (3, 1, 2)$ and $S = \{1, 2\}$. Then $M_2^\sigma(V) = 1$ and $M_2^{\sigma_{|S}}(V^S) = 10$, so the extended marginal vector corresponding to σ is not a pmas. (However, each of the five other extended marginal vectors does constitute a pmas.) ◁

Although in a marginal convex game not every extended marginal vector need be a pmas, it is still an open question whether for each marginal convex game a pmas *exists*.

Individual merge convexity does turn out to be sufficient for the extended marginal vector to be a pmas, as is shown in the following theorem.

Theorem 4.2.2 *Let $V \in NTU^N$ be an individual merge convex game and let $\sigma \in \Pi(N)$. Then $(M^{\sigma_{|S}}(V^S))_{S \subset N, S \neq \emptyset}$ is a pmas for V.*

Proof: First note that property (4.3) is satisfied by construction. For monotonicity condition (4.2), let $S \subset T \subset N, S \neq \emptyset$ and let $i \in S$. Define $\bar{S} = \{j \in S \,|\, \sigma^{-1}(j) < \sigma^{-1}(i)\}$ and $\bar{T} = \{j \in T \,|\, \sigma^{-1}(j) < \sigma^{-1}(i)\}$. We show that $M_i^{\sigma_{|S}}(V^S) \leq M_i^{\sigma_{|T}}(V^T)$ by distinguishing between three cases:

- If $\bar{S} = \bar{T}$, then $M_i^{\sigma_{|S}}(V^S) = M_i^{\sigma_{|T}}(V^T)$ by construction.
- Otherwise, if $\bar{S} = \emptyset$, then $M_i^{\sigma_{|S}}(V^S) = 0$ and by individual superadditivity, $M_i^{\sigma_{|T}}(V^T) \geq 0$.
- Otherwise, apply the individual merge property to i, $\bar{S}$ and $\bar{T}$. Because V is individual merge convex, the game V and all its subgames are also marginal convex. Since $M^{\sigma_{|S}}(V^S) \in C(V^S)$, there exists no $y \in V(\bar{S})$ such that $y > M_{\bar{S}}^{\sigma_{|S}}(V^S)$, so $M_{\bar{S}}^{\sigma_{|S}}(V^S)$ lies on or above the weak Pareto boundary of $V(\bar{S})$. Hence, there exists a $p \in WPar(\bar{S}) \cap IR(\bar{S})$ such that $M_{\bar{S}}^{\sigma_{|S}}(V^S) \geq p$. Taking $q = M_{\bar{T}}^{\sigma_{|T}}(V^T) \in V(\bar{T})$ and $r = M_{\bar{S}\cup\{i\}}^{\sigma_{|S}}(V^S) \in V(\bar{S}\cup\{i\})$, the individual merge property states that there exists an $s \in V(\bar{T} \cup \{i\})$ such that $s_i \geq M_i^{\sigma_{|S}}(V^S)$ and $s_{\bar{T}} \geq M_{\bar{T}}^{\sigma_{|T}}(V^T)$. The latter inequality together with the construction of a marginal vector imply $M_i^{\sigma_{|T}}(V^T) \geq s_i$ and hence, $M_i^{\sigma_{|S}}(V^S) \leq M_i^{\sigma_{|T}}(V^T)$.

From these three cases it follows that (4.2) is satisfied as well and hence, $(M^{\sigma_{|S}}(V^S))_{S \subset N, S \neq \emptyset}$ is a pmas for V. □

An immediate consequence of the TU equivalent of Theorem 4.2.2, as noted by Sprumont (1990), is that for convex TU games, the extended Shapley value is a pmas.

Using the MC value, as defined by (3.12), the result in Sprumont (1990) can be extended to the class of NTU games where the core of each subgame is a convex set.

Proposition 4.2.3 *Let $V \in NTU^N$ be an individual merge convex game such that $C(V^S)$ is convex for all $S \subset N, S \neq \emptyset$. Then $(MC(V^S))_{S \subset N, S \neq \emptyset}$ is a pmas for V.*

Proof: Because the core of every subgame of V is a convex set, the MC value equals the average of the marginal vectors in each subgame. Using Theorem 4.2.2, the assertion readily follows. □

Without the extra condition on $C(V^S)$, the extended MC value need not be a pmas.

As was the case for the extended marginal vectors, marginal convexity is not sufficient for the extended MC value to be a pmas, even if the core of every subgame is a convex set.

Example 4.2.2 Consider the 3-person game of Example 4.2.1. The MC value of the whole game equals $(5\frac{1}{6}, 3\frac{2}{3}, 2\frac{1}{6})$, while in the subgame consisting of players 1 and 2, the MC value equals $(\frac{1}{2}, 5)$. Monotonicity condition (4.2) is violated for player 2. ◁

4.3 Sequencing situations

One-machine sequencing situations were introduced in Curiel et al. (1989). Following the standard notions and notation of the ensuing literature (see, eg, the survey article of Borm et al. (2001)), there is a queue of players, each with one job, in front of a machine. Each player must have his job processed on this machine. The finite set of players is denoted by $N = \{1, ..., n\}$. The positions of the players in the queue are described by a bijection $\sigma : N \rightarrow \{1, \ldots, n\}$, where $\sigma(i) = j$ means that player i is at position j in the queue. The set of all such bijections is denoted by Π_N.[1] We assume that the initial order on the jobs before the processing of the machine starts is $\sigma_0 \in \Pi_N$, defined by $\sigma_0(i) = i$ for all $i \in N$. The processing time $p_i > 0$ of the job of player i is the time the machine takes to handle this job. For each player $i \in N$, the costs of spending time in the system is described by a cost function $k_i : \mathbb{R}_+ \rightarrow \mathbb{R}$, where $k_i(t)$ represents the costs of player i if his job is completed in t time units. Costs are assumed to be additive: the total costs of a coalition $S \subset N$ equal the sum of the individual costs of the members of S. Furthermore, the cost functions

[1] In sequencing it is more convenient to use these bijections rather than the orderings in $\Pi(N)$, which are defined the other way around.

are regular, ie, for all $i \in N$, $k_i(t)$ is increasing in t and $k_i(0) = 0$. A *sequencing situation* is described by a triple (N, p, k) with $p = (p_i)_{i \in N}$ and $k = (k_i)_{i \in N}$. We denote the class of all sequencing situations with player set N by SEQ^N.

We pay special attention to the class of sequencing situations with linear cost functions, ie, $k_i(t) = \alpha_i t$ for all $t \in \mathbb{R}_+$ with $\alpha_i \geq 0$. A *sequencing situation with linear cost functions* is denoted by (N, p, α) with $\alpha = (\alpha_i)_{i \in N}$. We denote all such sequencing situations with player set N by $LSEQ^N$.

The completion time $C(\sigma, i)$ of the job of player i if the jobs are processed (in a semi-active way) according to the order $\sigma \in \Pi_N$ is given by

$$C(\sigma, i) = \sum_{\{j \in N | \sigma(j) \leq \sigma(i)\}} p_j.$$

A processing order is called *semi-active* if there does not exist a job which could be processed earlier without altering the processing order, ie, if there are no unnecessary delays. The total costs of all players if the jobs are processed according to the order σ equal $\sum_{i \in N} k_i(C(\sigma, i))$. Clearly, because Π_N is finite, there exists an order for which total costs are minimised.

In the linear case, a processing order that minimises total costs of N is an order in which the jobs are processed in decreasing order with respect to the urgency index u_i defined by $u_i = \frac{\alpha_i}{p_i}$ (cf. Smith (1956)).

Example 4.3.1 Consider a linear one-machine sequencing situation $(N, p, \alpha) \in LSEQ^N$, where $N = \{1, 2, 3\}$, $p = (2, 2, 1)$ and $\alpha = (4, 6, 5)$. Then the urgencies for the players are $u_1 = 2$, $u_2 = 3$ and $u_3 = 5$, respectively. Hence, the optimal processing order is $(3, 2, 1)$ with total costs $5 \cdot 1 + 6 \cdot 3 + 4 \cdot 5 = 43$. $\triangleleft$

Note that an optimal order can be obtained from the initial order by consecutive switches of neighbours i, j with i directly in front of j and $u_i < u_j$. This process will be referred to as the Smith algorithm.

For a sequencing situation $(N, p, k) \in SEQ^N$ the costs $C_S(\sigma)$ of coalition S with respect to a processing order σ equal $C_S(\sigma) = \sum_{i \in S} k_i(C(\sigma, i))$. We want to determine the minimal costs of a coalition S when its members decide to cooperate. For this, we have to define which rearrangements of the coalition S are admissible with respect to the initial order. A bijection $\sigma \in \Pi_N$ is called *admissible* for S (cf. Curiel et al. (1989)) if it satisfies the following condition:

$$P(\sigma, j) = P(\sigma_0, j)$$

for all $j \in N \backslash S$, where for any $\tau \in \Pi_N$ the set of predecessors of a player $j \in N$ with respect to τ is defined as $P(\tau, j) = \{k \in N \mid \tau(k) \leq \tau(j)\}$.[2]

This condition implies, in particular, that the starting time of each player outside the coalition S is equal to his starting time in the initial order and the players of S are not allowed to "jump" over players outside S. The set of admissible orders for a coalition S is denoted by $\mathcal{A}(S)$.

We define the *sequencing game* (N, c) corresponding to the sequencing situation $(N, p, k) \in SEQ^N$ by

$$c(S) = \min_{\sigma \in \mathcal{A}(S)} \sum_{i \in S} k_i(C(\sigma, i)) \tag{4.4}$$

for all $S \subset N$. Contrary to the standard definition of a game, the coalitional value $c(S)$ reflects the *costs* of coalition S rather than its worth. As a result, some of the definitions in section 2.2 change accordingly.

In case the cost functions are linear, expression (4.4) can be rewritten in terms of $g_{ij} = \max\{0, \alpha_j p_i - \alpha_i p_j\}$, which equals the cost savings attainable by player i and j when i is directly in front of j, regardless of the exact position in the order. A coalition S is called *connected* with respect to σ if for all $i, j \in S$ and $\ell \in N$ such that $\sigma(i) < \sigma(\ell) < \sigma(j)$ it holds that $\ell \in S$. The Smith algorithm and (4.4) imply the following proposition (cf. Curiel et al. (1989)).

Proposition 4.3.1 *Let $(N, p, \alpha) \in LSEQ^N$ be a linear sequencing situation and let c be the corresponding sequencing game. Then for any coalition S that is connected with respect to σ_0 we have*

$$c(S) = \sum_{i \in S} \alpha_i C(\sigma_0, i) - \sum_{i,j \in S : i < j} g_{ij}.$$

For a coalition T that is not connected with respect to σ_0 the definition of admissible orders implies that

$$c(T) = \sum_{S \in T \backslash \sigma_0} c(S),$$

[2]This notion of admissibility is standard in the sequencing literature. Relaxations have been studied in Van Velzen and Hamers (2003) and Slikker (2003).

where $T\backslash\sigma_0$ is the set of components of T, a component of T being a maximally connected subset of T.

The *core* of a cost game c is defined by

$$C(c) = \{x \in \mathbb{R}^N \mid \sum_{i\in N} x_i = c(N), \forall_{S\subset N} : \sum_{i\in S} x_i \leq c(S)\}.$$

Analogous to benefit games, a core element is stable in the sense that if such a vector is proposed as cost allocation for the grand coalition, no coalition will have an incentive to split off and cooperate on their own.

A *sequencing rule* is a function $f : SEQ^N \to \mathbb{R}^N_+$ assigning to every sequencing situation $(N, p, k) \in SEQ^N$ a vector $f(N, p, k) \in \mathbb{R}^N_+$ such that $\sum_{i\in N} f_i(N, p, k) = c(N)$. A rule f is called *stable* if $f(N, p, k) \in C(c)$ for every sequencing situation $(N, p, k) \in SEQ^N$. In this chapter, we investigate one specific rule for the class of sequencing games with regular cost functions, the μ rule, which is the marginal vector corresponding to the initial order σ_0:

$$\mu_j(N, p, k) = c(P_j) - c(P_{j-1})$$

for all $j \in N$, where $P_j = P(\sigma_0, j) = \{1, \ldots, j\}$. In case the cost functions are linear, we can use Proposition 4.3.1 to rewrite this as

$$\mu_j(N, p, \alpha) = c(\{j\}) - \sum_{i\in N: i<j} g_{ij}.$$

According to this rule, the gain g_{ij} goes fully to player j, who is behind i in the queue.

Since every sequencing game is σ_0-component additive (cf. Curiel et al. (1995)), the μ rule is stable.[3] So letting the players at the front of the queue pay the highest costs and attributing the gains to the players at the back of the queue results in a stable outcome.

4.4 Drop out monotonicity

Suppose that one player in the queue decides to wait no longer and drops out. One natural question in this situation is how the costs of the other players will be affected by this. It seems natural that none of the players should be worse off if one

[3]For sequencing games arising from linear cost functions, concavity of these games can be used to establish stability of the μ rule (cf. Curiel et al. (1989)).

of them drops out of the queue. Formally, a rule $f : SEQ^N \to \mathbb{R}^N_+$ is called *drop out monotonic* if for all sequencing situations $(N, p, k) \in SEQ^N$ and all $q \in N$ we have

$$f_j(N, p, k) \geq f_j((N, p, k)^{-q})$$

for all $j \in N\backslash\{q\}$, where $(N, p, k)^{-q} = (N\backslash\{q\}, (p_i)_{i \in N\backslash\{q\}}, (k_i)_{i \in N\backslash\{q\}})$ and the initial order in $(N, p, k)^{-q}$ is σ_0 restricted to $N\backslash\{q\}$.

Proposition 4.4.1 *μ is drop out monotonic on the class of sequencing situations with linear cost functions.*

Proof: Let $(N, p, \alpha) \in LSEQ^N$ be a sequencing situation with linear cost functions, let $q \in N$ and let $j \in N\backslash\{q\}$. If $j < q$, then $\mu_j(N, p, \alpha) = c(\{j\}) - \sum_{i \in N: i < j} g_{ij} = \mu_j((N, p, \alpha)^{-q})$. If $j > q$, then $\mu_j((N, p, \alpha)^{-q}) = (\sum_{i=1}^{j} p_i - p_q)\alpha_j - \sum_{i \in N: i<j} g_{ij} + g_{qj} = ((\sum_{i=1}^{j} p_i)\alpha_j - \sum_{i \in N: i<j} g_{ij}) + (g_{qj} - p_q\alpha_j) = \mu_j(N, p, \alpha) - \min\{p_j\alpha_q, p_q\alpha_j\} \leq \mu_j(N, p, \alpha)$. □

Proposition 4.4.1 shows that the μ rule is drop out monotonic in case the cost functions are linear. The question now arises whether this is the only rule satisfying this property. In the following theorem, we show that within the class of sequencing situations with regular cost functions (not necessarily linear), the μ rule is the only possible stable and drop out monotonic rule.

Theorem 4.4.2 *Let f be a rule on the class of sequencing situations with regular cost functions. If f is stable and drop out monotonic, then f equals the μ rule.*

Proof: Let $(N, p, k) \in SEQ^N$ be a sequencing situation with regular cost functions and let f be a stable and drop out monotonic rule. Denote the corresponding sequencing game by c and denote $f_i^S = f_i(S, (p_j)_{j \in S}, (k_j)_{j \in S})$ and $\mu_i = \mu_i(N, p, k)$ for all $i \in N$ and $S \subset N, S \neq \emptyset$. We show that $f = \mu$ by an inductive argument.
First, from drop out monotonicity it follows that $f_1^N \geq f_1^{\{1\}}$. From stability we have $f_1^N \leq c(\{1\}) = f_1^{\{1\}}$. Hence, $f_1^N = f_1^{\{1\}} = c(\{1\}) = \mu_1$.
Next, let $j \in \{2, \ldots n\}$. Assume that $f_i^N = f_i^{P_{j-1}} = \mu_i$ for all $i \in P_{j-1}$. From drop out monotonicity we have $f_i^N \geq f_i^{P_j}$ for all $i \in P_j$, so $\sum_{i \in P_j} f_i^N \geq \sum_{i \in P_j} f_i^{P_j} = c(P_j)$. By stability, $\sum_{i \in P_j} f_i^N \leq c(P_j)$. So, $\sum_{i \in P_j} f_i^N = c(P_j)$ and, using the induction hypothesis, $f_j^N = c(P_j) - \sum_{i \in P_{j-1}} f_i^N = c(P_j) - c(P_{j-1}) = \mu_j$.
Hence, we conclude that $f = f^N = \mu$. □

It follows from Proposition 4.4.1 and Theorem 4.4.2 that drop out monotonicity and stability together characterise the μ rule on the class of sequencing situations with linear cost functions.

Theorem 4.4.2 states that if the cost functions are regular, then there exists at most one stable drop out monotonic rule, which must be the μ rule. However, not for every class of regular cost functions the μ rule is drop out monotonic. It is readily seen that for concave cost functions, drop out monotonicity of μ does not hold. Hence, for such situations no stable drop out monotonic rule exists.

For convex cost functions, the next proposition establishes drop out monotonicity of μ in case there are three players involved.

Proposition 4.4.3 *μ is drop out monotonic on the class of 3-player sequencing situations with convex cost functions.*

Proof: Let $(N,p,k) \in SEQ^N$ be such a situation. To avoid unnecessarily complicated notation, we only show that $\mu_3 \geq \mu_3^{-1}$, where $\mu_3 = \mu_3(N,p,k)$ and $\mu_3^{-1} = \mu_3(\{2,3\},(p_2,p_3),(k_2,k_3))$. The other cases can be shown in a similar way. Let $\hat{\sigma}$ be an optimal order for N. Then, denoting by $\hat{\sigma}_S$ the order on the players in S induced by $\hat{\sigma}$,

$$\begin{aligned}
\mu_3 &= \sum_{i\in N} k_i(C(\hat{\sigma},i)) - \min_{\sigma\in\Pi(\{1,2\})} \sum_{i\in\{1,2\}} k_i(C(\sigma,i)) \\
&\geq \sum_{i\in N} k_i(C(\hat{\sigma},i)) - \sum_{i\in\{1,2\}} k_i(C(\hat{\sigma}_{\{1,2\}},i)) \\
&= k_1(C(\hat{\sigma},1)) - k_1(C(\hat{\sigma}_{\{1,2\}},1)) + k_2(C(\hat{\sigma},2)) - k_2(C(\hat{\sigma}_{\{1,2\}},2)) + \\
&\quad\ k_3(C(\hat{\sigma},3)) \\
&\geq 0 + k_2(C(\hat{\sigma}_{\{2,3\}},2)) - k_2(p_2) + k_3(C(\hat{\sigma}_{\{2,3\}},3)) \\
&\geq \min_{\sigma\in\Pi(\{2,3\})} \sum_{i\in\{2,3\}} k_i(C(\sigma,i)) - k_2(p_2) \\
&= \mu_3^{-1},
\end{aligned}$$

where the second inequality follows from regularity and convexity of the cost functions. □

A type of cost function that has been studied in the literature (cf. Moore (1968)) arises when the players face a due date. Suppose player $i \in N$ must have his job completed before a certain due date $d_i \geq 0$. If he manages to do this, waiting is costless and if he is tardy, he has to pay a penalty of one unit. So,

$$k_i(t) = \begin{cases} 1 & \text{if } t > d_i, \\ 0 & \text{if } t \leq d_i. \end{cases} \tag{4.5}$$

Finding an optimal order for these cost functions boils down to minimising the number of tardy jobs. An efficient algorithm for this is provided by Moore (1968). The following proposition is immediate.

Proposition 4.4.4 *The μ rule is drop out monotonic on the class of sequencing situations with cost functions as in (4.5).*

4.5 Sequencing with a single due date

In this section, we consider a mixture of the two main types of cost functions, linear cost functions and step functions as in (4.5), that were analysed in the previous section. One of the players, $i^* \in N$, faces a due date, and the players in $N \backslash \{i^*\}$ have linear cost functions. Player i^* also has a linear cost parameter α_{i^*}, but if his job is completed after a certain due date d, he also has to pay a fixed penalty $\gamma > 0$.

Note that the cost functions in this framework are regular, so we can apply Theorem 4.4.2 to conclude that there is at most one stable and drop out monotonic rule, which must be the μ rule. Contrary to the situation with only linear cost functions, however, the μ rule may not be drop out monotonic, as is shown in the next example.

Example 4.5.1 Consider the sequencing situation $(N, p, k) \in SEQ^N$ with $N = \{1, 2, 3\}$, processing times $p_1 = 1$, $p_2 = 5$ and $p_3 = 1$ and cost functions $k_1(t) = 1000t$,

$$k_2(t) = \begin{cases} 100 & \text{if } t > 5, \\ 0 & \text{if } t \leq 5, \end{cases}$$

and $k_3(t) = 10t$. For the grand coalition, the optimal order is $(1, 3, 2)$ with costs 1000+100+20=1120, while for coalition $\{1, 2\}$, the optimal order is $(1, 2)$ with costs 1000+100=1100. So, according to the μ rule, player 3 should pay 20.

Now, consider the situation in which player 1 has dropped out of the queue. The optimal order for coalition $\{2, 3\}$ is then $(2, 3)$ with costs 0+60=60 and the costs for coalition $\{2\}$ equal 0. So, in this reduced situation, player 3 has to pay 60 according to the μ rule, which is more than in the original situation. Hence, the μ rule is not drop out monotonic. $\triangleleft$

In the remainder of this section, we restrict ourselves to sequencing situations with a single due date and equal processing times (which without loss of generality we assume to be 1). It turns out that for this class of situations, drop out monotonicity of the μ rule can be established.

A *sequencing situation with a single due date* is represented by a 5-tuple $(N, \alpha, i^*, d, \gamma)$ with player set $N = \{1, \ldots, n\}$, a vector of cost parameters $\alpha \in \mathbb{R}_+^N$, player $i^* \in N$ facing the due date $d \in \mathbb{N}$ and penalty $\gamma > 0$. Such a situation is a sequencing situation $(N, p, k) \in SEQ^N$ with $p_i = 1$ for all $i \in N$, $k_i(t) = \alpha_i t$ for all $i \in N \backslash \{i^*\}$ and

$$k_{i^*}(t) = \begin{cases} \alpha_{i^*} t & \text{if } t \leq d, \\ \alpha_{i^*} t + \gamma & \text{if } t > d. \end{cases}$$

Let $(N, \alpha, i^*, d, \gamma)$ be a sequencing situation with a single due date. With this situation we associate two games, $\hat{c}$ and c. The game $\hat{c}$ is the formal sequencing game defined by applying (4.4) to the actual cost functions $(k_i)_{i \in N}$. The game c is the auxiliary sequencing game that arises if γ is set to 0, ie, the game corresponding to the linear sequencing situation $(N, p, \alpha) \in LSEQ^N$ with $p_i = 1$ for all $i \in N$.

Games arising from linear sequencing situations are concave, ie, the reverse inequalities hold in (3.1)-(3.3). The single due date game $\hat{c}$ need not be concave, as is shown in the following example.

Example 4.5.2 Consider $(N, \alpha, i^*, d, \gamma)$ with $N = \{1, 2, 3\}$, $\alpha = (10, 2, 5)$, $i^* = 2$, $d = 1$ and $\gamma = 10$. Then $\hat{c}(N) - \hat{c}(\{1, 2\}) = 37 - 22 > 26 - 14 = \hat{c}(\{2, 3\}) - \hat{c}(\{2\})$. $\triangleleft$

Let $j \in N$ and let $\sigma_j \in \Pi_{P_j}$ be the unique urgency order on P_j (recall $P_j = \{1, \ldots, j\}$), where ties are broken by some fixed order on the players, starting with i^*. Then the following proposition is immediate.

Proposition 4.5.1 *The optimal order on P_j is either σ_j or the order in which the completion time of job i^* is exactly the due date and all other jobs are ordered in decreasing urgency.*

In order to prove drop out monotonicity of the μ rule, we need to introduce some auxiliary lemmas and notation. By β_j we denote the cost of making player i^* in time, starting from the urgency order σ_j, so

$$\beta_j = \begin{cases} \displaystyle\sum_{i \in N: d \leq \sigma_j(i) < \sigma_j(i^*)} \alpha_i - (\sigma_j(i^*) - d)\alpha_{i^*} & \text{if } d < \sigma_j(i^*) \text{ and } i^* \leq j, \\ 0 & \text{if } \sigma_j(i^*) \leq d \text{ or } i^* > j. \end{cases}$$

Note that $\beta_j \geq 0$. From Proposition 4.5.1 it follows that

$$\hat{c}(P_j) = c(P_j) + A_j,$$

where $A_j = \min\{\gamma, \beta_j\}$ represents the extra costs as a result of the due date.

For $q \in N \backslash \{j\}$, we denote by β_j^{-q} and A_j^{-q} the corresponding costs in the situation where player q has dropped out of the queue (for $q = i^*$, we define $\beta_j^{-i^*} = 0$). Next, we define $\mu_j = \mu_j(N, p, \alpha) = c(P_j) - c(P_{j-1})$ and $\hat{\mu}_j = \mu_j(N, p, k) = \hat{c}(P_j) - \hat{c}(P_{j-1})$ and μ_j^{-q} and $\hat{\mu}_j^{-q}$ accordingly.

From Proposition 4.4.1 we have $\mu_j^{-q} \leq \mu_j$. To show that the μ rule is drop out monotonic on the class of sequencing situations with a single due date, we have to show that $\hat{\mu}_j^{-q} \leq \hat{\mu}_j$ for all $j \neq q$. If player i^* drops out of the queue, then it is readily seen that this inequality is satisfied, so in the remainder we assume that $q \neq i^*$. Rewriting the inequality, we have to show that for all $q \in N \backslash \{i^*, j\}$,

$$A_j - A_{j-1} - (A_j^{-q} - A_{j-1}^{-q}) \geq -\min\{\alpha_j, \alpha_q\}, \tag{4.6}$$

where the right hand side equals $\mu_j^{-q} - \mu_j$ by the proof of Proposition 4.4.1.

First of all, we establish (4.6) for some easy cases.

Lemma 4.5.2 *If $q > j$ or $i^* > j$, then $A_j - A_{j-1} - (A_j^{-q} - A_{j-1}^{-q}) \geq 0 \geq -\min\{\alpha_j, \alpha_q\}$.*

Proof: If $q > j$, then $A_j^{-q} = A_j$ and $A_{j-1}^{-q} = A_{j-1}$. If $i^* > j$, then $A_j = A_{j-1} = A_j^{-q} = A_{j-1}^{-q} = 0$. From this the assertion follows. □

For $i^* \leq j, q < j$ we have the following expressions:

$$\beta_j = \begin{cases} \displaystyle\sum_{\substack{i \in N: \\ d \leq \sigma_j(i) < \sigma_j(i^*)}} \alpha_i - (\sigma_j(i^*) - d)\alpha_{i^*} & \text{if } d < \sigma_j(i^*), \\ 0 & \text{if } \sigma_j(i^*) \leq d. \end{cases}$$

$$\beta_j^{-q} = \begin{cases} \beta_j & \text{if } d < \sigma_j(i^*) < \sigma_j(q), \\ \sum\limits_{\substack{i \in N: \\ d \le \sigma_j(i) < \sigma_j(i^*)}} \alpha_i - \alpha_q - (\sigma_j(i^*) - d - 1)\alpha_{i^*} & \text{if } d < \sigma_j(q) < \sigma_j(i^*), \\ \sum\limits_{\substack{i \in N: \\ d < \sigma_j(i) < \sigma_j(i^*)}} \alpha_i - (\sigma_j(i^*) - d - 1)\alpha_{i^*} & \text{if } \sigma_j(q) \le d < \sigma_j(i^*), \\ 0 & \text{if } \sigma_j(i^*) \le d. \end{cases}$$

$$\beta_{j-1} = \begin{cases} \beta_j & \text{if } d < \sigma_j(i^*) < \sigma_j(j), \\ \sum\limits_{\substack{i \in N: \\ d \le \sigma_j(i) < \sigma_j(i^*)}} \alpha_i - \alpha_j - (\sigma_j(i^*) - d - 1)\alpha_{i^*} & \text{if } d < \sigma_j(j) < \sigma_j(i^*), \\ \sum\limits_{\substack{i \in N: \\ d < \sigma_j(i) < \sigma_j(i^*)}} \alpha_i - (\sigma_j(i^*) - d - 1)\alpha_{i^*} & \text{if } \sigma_j(j) \le d < \sigma_j(i^*), \\ 0 & \text{if } \sigma_j(i^*) \le d \end{cases}$$

$$\beta_{j-1}^{-q} = \begin{cases} \beta_{j-1} & \text{if } \ d < \sigma_j(i^*) < \sigma_j(q), \\ \beta_j^{-q} & \text{if } \ d < \sigma_j(i^*) < \sigma_j(j), \\ & \quad\ \ \sigma_j(q) < \sigma_j(i^*), \\ \sum\limits_{\substack{i \in N: \\ d \le \sigma_j(i) < \sigma_j(i^*)}} \alpha_i - \alpha_j - \alpha_q - (\sigma_j(i^*) - d - 2)\alpha_{i^*} & \text{if } \ d < \sigma_j(q) < \sigma_j(i^*), \\ & \quad\ \ d < \sigma_j(j) < \sigma_j(i^*), \\ \sum\limits_{\substack{i \in N: \\ d < \sigma_j(i) < \sigma_j(i^*)}} \alpha_i - \alpha_q - (\sigma_j(i^*) - d - 2)\alpha_{i^*} & \text{if } \ \sigma_j(j) \le d < \sigma_j(q) < \sigma_j(i^*), \\ \sum\limits_{\substack{i \in N: \\ d < \sigma_j(i) < \sigma_j(i^*)}} \alpha_i - \alpha_j - (\sigma_j(i^*) - d - 2)\alpha_{i^*} & \text{if } \ \sigma_j(q) \le d < \sigma_j(j) < \sigma_j(i^*), \\ \sum\limits_{\substack{i \in N: \\ d+2 \le \sigma_j(i) < \sigma_j(i^*)}} \alpha_i - (\sigma_j(i^*) - d - 2)\alpha_{i^*} & \text{if } \ \sigma_j(q) \le d, \\ & \quad\ \ \sigma_j(j) \le d, \\ & \quad\ \ d + 2 \le \sigma_j(i^*), \\ 0 & \text{if } \ \sigma_j(i^*) \le d \text{ or} \\ & \quad\ \ \sigma_j(q) \le d, \\ & \quad\ \ \sigma_j(j) \le d, \\ & \quad\ \ \sigma_j(i^*) = d + 1. \end{cases}$$

Lemma 4.5.3 *If $i^* \le j$ and $q < j$, then $\beta_j \ge \beta_j^{-q}$.*

Proof: Assume that $i^* \le j, q < j$. Distinguish between the following four cases:

a) $\sigma_j(i^*) < \sigma_j(q)$. Then $\beta_j = \beta_j^{-q}$.

b) $d < \sigma_j(q) < \sigma_j(i^*)$. Then $\beta_j - \beta_j^{-q} = \alpha_q - \alpha_{i^*} \geq 0$.

c) $\sigma_j(q) \leq d < \sigma_j(i^*)$. Then $\beta_j - \beta_j^{-q} = \alpha_{\sigma_j^{-1}(d)} - \alpha_{i^*} \geq 0$.

d) $\sigma_j(q) < \sigma_j(i^*) \leq d$. Then $\beta_j = \beta_j^{-q} = 0$.

□

As an immediate corollary, we have $A_j \geq A_j^{-q}$ for $i^* \leq j, q < j$. The case $j = i^*$ is considered in the following lemma.

Lemma 4.5.4 *If $q < j$, then $A_{i^*} - A_{i^*-1} - (A_{i^*}^{-q} - A_{i^*-1}^{-q}) \geq 0 \geq -\min\{\alpha_j, \alpha_q\}$.*

Proof: Obviously, $A_{i^*-1} = A_{i^*-1}^{-q} = 0$, so the assertion follows from Lemma 4.5.3. □

As a result of the previous lemma, we only consider the case $i^* < j$ in the remainder.

Lemma 4.5.5 *If $i^* < j, q < j$ and $\alpha_j \geq \alpha_q$, then $\beta_j^{-q} \geq \beta_{j-1}$.*

Proof: Assume that $i^* < j, q < j$ and $\alpha_j > \alpha_q$, then $\sigma_j(j) < \sigma_j(q)$. (The proof for $\alpha_j = \alpha_q$ is similar.) Distinguish between the following seven cases:

a) $d < \sigma_j(i^*) < \sigma_j(j) < \sigma_j(q)$. Then $\beta_j^{-q} = \beta_{j-1} = 0$.

b) $d < \sigma_j(j) < \sigma_j(i^*) < \sigma_j(q)$. Then $\beta_j^{-q} - \beta_{j-1} = \alpha_j - \alpha_{i^*} \geq 0$.

c) $\sigma_j(j) \leq d < \sigma_j(i^*) < \sigma_j(q)$. Then $\beta_j^{-q} - \beta_{j-1} = \alpha_{\sigma_j^{-1}(d)} - \alpha_{i^*} \geq 0$.

d) $d < \sigma_j(j) < \sigma_j(q) < \sigma_j(i^*)$. Then $\beta_j^{-q} - \beta_{j-1} = \alpha_j - \alpha_q \geq 0$.

e) $\sigma_j(j) \leq d < \sigma_j(q) < \sigma_j(i^*)$. Then $\beta_j^{-q} - \beta_{j-1} = \alpha_{\sigma_j^{-1}(d)} - \alpha_{i^*} \geq 0$.

f) $\sigma_j(j) < \sigma_j(q) \leq d < \sigma_j(i^*)$. Then $\beta_j^{-q} - \beta_{j-1} = 0$.

g) $\sigma_j(i^*) \leq d$. Then $\beta_j^{-q} = \beta_{j-1} = 0$.

□

Similarly, one can prove the following lemma.

Lemma 4.5.6 *If $i^* < j, q < j$ and $\alpha_j \leq \alpha_q$, then $\beta_{j-1} \geq \beta_j^{-q}$.*

In Lemmas 4.5.7 and 4.5.8, we establish (4.6) for the cases not covered in Lemmas 4.5.2 and 4.5.4.

Lemma 4.5.7 *If $i^* < j, q < j$ and $\alpha_j \geq \alpha_q$, then $A_j - A_{j-1} - (A_j^{-q} - A_{j-1}^{-q}) \geq -\alpha_q$.*

Proof: Assume that $i^* < j, q < j$ and $\alpha_j \geq \alpha_q$. It follows from Lemmas 4.5.3 and 4.5.5 that $\beta_j \geq \beta_j^{-q} \geq \beta_{j-1} \geq \beta_{j-1}^{-q}$. Define $\ell = A_j - A_{j-1} - (A_j^{-q} - A_{j-1}^{-q})$. Distinguish between the following five cases:

a) $\gamma \leq \beta_{j-1}^{-q}$. Then $\ell = 0$.

b) $\beta_{j-1}^{-q} < \gamma \leq \beta_{j-1}$. Then $\ell = \beta_{j-1}^{-q} - \gamma$.

c) $\beta_{j-1} < \gamma \leq \beta_j^{-q}$. Then $\ell = -\beta_{j-1} + \beta_{j-1}^{-q}$.

d) $\beta_j^{-q} < \gamma \leq \beta_j$. Then $\ell = \gamma - \beta_{j-1} - (\beta_j^{-q} - \beta_{j-1}^{-q})$.

e) $\beta_j < \gamma$. Then $\ell = \beta_j - \beta_{j-1} - (\beta_j^{-q} - \beta_{j-1}^{-q})$.

From these five cases it follows that $\ell \geq -\beta_{j-1} + \beta_{j-1}^{-q}$, so it suffices to show $\beta_{j-1} - \beta_{j-1}^{-q} \leq \alpha_q$. Assume that $\alpha_j > \alpha_q$, then $\sigma_j(j) < \sigma_j(q)$. (The proof for $\alpha_j = \alpha_q$ is similar.) Distinguish between the following six cases:

a) $\sigma_j(i^*) < \sigma_j(q)$. Then $\beta_{j-1} = \beta_{j-1}^{-q}$.

b) $d < \sigma_j(j) < \sigma_j(q) < \sigma_j(i^*)$. Then $\beta_{j-1} - \beta_{j-1}^{-q} = \alpha_q - \alpha_{i^*} \leq \alpha_q$.

c) $\sigma_j(j) \leq d < \sigma_j(q) < \sigma_j(i^*)$. Then $\beta_{j-1} - \beta_{j-1}^{-q} = \alpha_q - \alpha_{i^*} \leq \alpha_q$.

d) $\sigma_j(q) \leq d \leq \sigma_j(i^*) - 2$. Then $\beta_{j-1} - \beta_{j-1}^{-q} = \alpha_{\sigma_j^{-1}(d+1)} - \alpha_{i^*} \leq \alpha_q$.

e) $\sigma_j(q) \leq d = \sigma_j(i^*) - 1$. Then $\beta_{j-1} = \beta_{j-1}^{-q}$.

f) $\sigma_j(i^*) \leq d$. Then $\beta_{j-1} = \beta_{j-1}^{-q} = 0$.

□

Similarly, one can prove the following lemma, using Lemma 4.5.6.

Lemma 4.5.8 *If* $i^* < j, q < j$ *and* $\alpha_q \geq \alpha_j$*, then* $A_j - A_{j-1} - (A_j^{-q} - A_{j-1}^{-q}) \geq -\alpha_j$.

From Lemmas 4.5.2, 4.5.4, 4.5.7 and 4.5.8 one readily concludes the following theorem.

Theorem 4.5.9 μ *is drop out monotonic on the class of sequencing situations with a single due date.*

Since the resulting sequencing games are σ_0-component additive, the μ rule also satisfies stability (cf. Curiel et al. (1995)). As a result, the μ rule is the unique stable and drop out monotonic rule on this class of situations.

Chapter 5

Communication

5.1 Introduction

In cooperative game theory the central question is how to divide the value of the grand coalition in a fair way, given the values of all subcoalitions. The value of a coalition is interpreted as the (monetary) amount the members of that coalition can obtain if they cooperate. Often, however, this hypothetical maximum is based on some simplifying assumptions on the underlying problem. Eg, in linear production situations (cf. Owen (1975)), it is assumed that all the players in a coalition are physically able to pool their resources. But one can imagine that as a result of transportation difficulties, cooperation between certain players is restricted.

Myerson (1977) models such a problem as a communication situation, which consists of an underlying game (eg, linear production game) and an undirected graph representing the players' communication possibilities (eg, transport routes). A communication situation gives rise to a graph-restricted game, in which the value of a coalition of players reflects their underlying theoretical possibilities as well as their ability to realise them. A recent overview of communication situations and related models is provided by Slikker and Van den Nouweland (2001).

The literature on communication situations mainly focuses on the case in which the underlying game is a transferable utility game, which then gives rise to a TU graph-restricted game. In section 3, we indicate a disadvantage of modelling the communication restrictions in this way. To address this, in this chapter, which is based on Hendrickx (2002), we consider nontransferable utility communication situations and compare the two approaches.

Myerson (1977) proposes the Shapley value of the graph-restricted game (later called the Myerson value) as solution concept for TU communication situations. We use the MC value, which is an NTU generalisation of the Shapley value (cf. Otten et al. (1998)), to extend the Myerson value to the class of NTU communication situations.

Van den Nouweland and Borm (1991) and Slikker (2000) study the inheritance of properties in TU communication situations, ie, given a certain property of TU games, they provide necessary and sufficient conditions which a graph must satisfy such that for every game satisfying that property, the graph-restricted game satisfies the same property. We extend their analysis to NTU communication situations and relate the TU and NTU models.

This chapter is organised as follows. In section 2, we introduce some notation and basic definitions. In section 3, we define graph-restricted games, discuss the models of TU and NTU communication situations and extend the Myerson value. Finally, in section 4, inheritance of properties is analysed.

5.2 Notation and definitions

For a finite set S, the comprehensive convex hull of a set $A \subset \mathbb{R}^S$ is defined by

$$cc(A) = \{x \in \mathbb{R}^S \mid \exists_{t\in\mathbb{N}}\exists_{x^1,\dots,x^t\in A}\exists_{(\lambda_1,\dots,\lambda_t)\in\Delta^t} : x \leq \sum_{i=1}^{t} \lambda_i x^i\},$$

where $\Delta^t = \{\lambda \in \mathbb{R}^t_+ \mid \sum_{i=1}^t \lambda_i = 1\}$.

For $x \in \mathbb{R}$ we define

$$Z^{S,x} = \{y \in \mathbb{R}^S \mid \sum_{i\in S} y_i \leq x\}$$

and

$$\bar{Z}^{S,x} = \{y \in \mathbb{R}^S \mid \sum_{i\in S} y_i \leq x, \forall_{i\in S} : y_i \leq x\}.$$

Using this notation, a TU game $v \in TU^N$ gives rise to an NTU game $V \in NTU^N$ by defining

$$V(S) = Z^{S,v(S)} \tag{5.1}$$

for all $S \subset N, S \neq \emptyset$.

A *communication network* is an undirected graph (N, E), where the vertices $N = \{1, \ldots, n\}$ represent the players and the edges $E \subset \{\{i, j\} \mid i, j \in N, i \neq j\}$ represent the (bilateral) communication links between the players.

Let (N, E) be a communication network. For all $S \subset N$ we define

$$E(S) = \{\{i, j\} \in E \mid i, j \in S\},$$

the set of links between members of S.

A *path* in (N, E) is a sequence of players $(x_1, \ldots, x_t)$ such that $\{x_i, x_{i+1}\} \in E$ for all $i \in \{1, \ldots, t-1\}$. A *cycle* is a path $(x_1, \ldots, x_t)$ where $t \geq 4$, $x_t = x_1$ and $x_1, \ldots, x_{t-1}$ are all distinct points. Two players $i, j \in N, i \neq j$ are *connected* if there exists a path $(x_1, \ldots, x_t)$ with $x_1 = i$ and $x_t = j$.

A network (N, E) is called

- *empty* if $E = \emptyset$;
- *complete* if $E = \{\{i, j\} \mid i, j \in N, i \neq j\}$;
- *connected* if each pair $i, j \in N, i \neq j$ is connected;
- *cycle-free* if it does not contain a cycle;
- *cycle-complete* if for every cycle $(x_1, \ldots, x_t)$, $\{x_i, x_j\} \in E$ for all $i, j \in \{1, \ldots, t\}, i \neq j$;
- a *star* if there exists an $i \in N$ such that $E = \{\{i, j\} \mid j \in N \backslash \{i\}\}$.

For $S \subset N$ we denote the *components* of S with respect to (N, E) by S/E, ie, $S/E = \{S_1, \ldots, S_m\}$ such that

- $(S_i, E(S_i))$ is connected for all $i \in \{1, \ldots, m\}$;
- $S_i \cap S_j = \emptyset$ for all $i, j \in \{1, \ldots, m\}, i \neq j$;
- $(S_i \cup S_j, E(S_i \cup S_j))$ is not connected for all $i, j \in \{1, \ldots, m\}, i \neq j$;
- $S_i \neq \emptyset$ for all $i \in \{1, \ldots, m\}$.

Player $i \in N$ is called a *dummy player* in the game $V \in NTU^N$ if

$$V(S \cup \{i\}) = V(S) \times V(\{i\})$$

for all $S \subset N \backslash \{i\}, S \neq \emptyset$.

To avoid confusion, we denote the set of weak Pareto optimal allocations for a coalition S in a game $V \in NTU^N$ by $WPar(V, S)$ rather than $WPar(S)$ throughout this chapter. Similarly, we write $IR(V, S)$ instead of $IR(S)$.

To round off this section, a *TU communication situation* is a triple (N, v, E), where $v \in TU^N$ is an underlying TU game and (N, E) is a communication network with the same player set. Similarly, an *NTU communication situation* is a triple (N, V, E) with $V \in NTU^N$. The classes of TU communication situatons and NTU communication situations with player set N are denoted by TUC^N and $NTUC^N$, respectively.

5.3 Graph-restricted games

In this section, we define graph-restricted games, starting with TU games. We point out why TU graph-restricted games might not be a satisfactory way of modelling the role of the communication restrictions. To address this, we define NTU graph-restricted games and compare the two models.

Let $(N, v, E) \in TUC^N$. The game $v \in TU^N$ represents the underlying possibilities of the players. However, these possibilities cannot come all to fruition because of the communication restrictions represented by the network (N, E). The *graph-restricted game* $v^E \in TU^N$ (cf. Myerson (1977)) takes these restrictions into account by considering the values of the components that can communicate:

$$v^E(S) = \sum_{C \in S/E} v(C) \tag{5.2}$$

for all $S \subset N$.

The resulting graph-restricted game is again a TU game and hence, side payments between the players through binding contracts are assumed to be possible, even between players that cannot communicate. When solving the subsequent graph-restricted game, this possibility should be carefully taken into account, eg, by only considering component decomposable solution concepts. A solution f is *component decomposable* (cf. Myerson (1977)) if applying f to a communication situation and applying f to all its components separately leads to the same outcome for each player.

Another way to address this element of the model is to exclude side payments between noncommunicating players *a priori* and consider NTU games. So, let

$(N, V, E) \in NTUC^N$. The graph-restricted game $V^E \in NTU^N$ (cf. Slikker and Van den Nouweland (2001)) is defined by

$$V^E(S) = \prod_{C \in S/E} V(C) \tag{5.3}$$

for all $S \subset N, S \neq \emptyset$. Note that the graph-restricted game V^E is again an element of NTU^N, satisfying all the basic assumptions as stated in section 2.3.

In particular, NTU graph-restricted games can be constructed for NTU games that arise from TU games (as in (5.1)). So, a TU communication situation $(N, v, E) \in TUC^N$ gives rise to two graph-restricted games: v^E and V^E.[1] But whereas side payments between players that cannot communicate are still possible in v^E, they are ruled out in V^E. In the remainder of this section, we study the relation between these two graph-restricted games.

The difference between the two games is illustrated in the following example.

Example 5.3.1 Consider the communication situation $(N, v, E) \in TUC^N$ with $N = \{1, 2, 3\}$, $v(S) = 1$ for all $S \subset N, |S| \geq 2$ and $E = \{\{1, 2\}\}$, so players 1 and 2 can communicate, while player 3 cannot communicate with either of them.

In the TU graph-restricted game v^E, the value of the grand coalition equals

$$v^E(N) = v(\{1, 2\}) + v(\{3\}) = 1 + 0 = 1,$$

while in the NTU graph-restricted game V^E,

$$V^E(N) = V(\{1, 2\}) \times V(\{3\}) = \{x \in \mathbb{R}^N \mid x_1 + x_2 \leq 1, x_3 \leq 0\}.$$

So, whereas in the TU graph-restricted game $(0, 0, 1)$ is a feasible (though unattractive) payoff vector, in the NTU graph-restricted game a positive payoff to player 3 is ruled out *ex ante*. $\triangleleft$

Although the two graph-restricted games need not be the same, they have the same core, as is shown in the following proposition.

Proposition 5.3.1 *Let $(N, v, E) \in TUC^N$. Then $C(v^E) = C(V^E)$.*

[1] By V^E we denote the game that is obtained from v by first taking the corresponding game V (as in (5.1)) and then applying (5.3). Reversing the order, ie, first applying (5.2) and then taking the corresponding NTU game would result in a different graph-restricted game, which does not essentially differ from the TU restricted game v^E.

Proof: "$\subset$" Let $x \in C(v^E)$. Because $\sum_{i\in N} x_i = v^E(N)$ and $v^E(N) = \sum_{C\in N/E} v(C)$, we have $\sum_{C\in N/E}\sum_{i\in C} x_i = \sum_{C\in N/E} v(C)$. Furthermore, $\sum_{i\in C} x_i \geq v^E(C) = v(C)$ for all $C \in N/E$, so $\sum_{i\in C} x_i = v(C)$ for all $C \in N/E$. From this it follows that $x \in V^E(N)$. Next, let $S \subset N, S \neq \emptyset$. Then $V^E(S) = \prod_{C\in S/E} V(C) = \prod_{C\in S/E} Z^{C,v(C)} \subset Z^{S,\sum_{C\in S/E} v(C)}$. Since $\sum_{i\in S} x_i \geq v^E(S) = \sum_{C\in S/E} v(C)$, there can exist no $y \in V^E(S)$ such that $y > x_S$. Hence, $x \in C(V^E)$.
"$\supset$" Let $x \in C(V^E)$. For all $C \in N/E$, $\nexists_{y\in V^E(C)} : y > x_C$ implies $\sum_{i\in C} x_i = v^E(C) = v(C)$. Hence, $\sum_{i\in N} x_i = \sum_{C\in N/E}\sum_{i\in C} x_i = \sum_{C\in N/E} v(C) = v^E(N)$. Next, let $S \subset N, S \neq \emptyset$. Then for all $C \in S/E$, $\nexists_{y\in V^E(C)} : y > x_C$ implies $\sum_{i\in C} x_i \geq v(C)$, and hence, $\sum_{i\in S} x_i = \sum_{C\in S/E}\sum_{i\in C} x_i \geq \sum_{C\in S/E} v(C) = v^E(S)$. Hence, $x \in C(v^E)$. □

Not only do the cores of the two graph-restricted games coincide, but also the Shapley/MC solutions. To show this, we first prove equality between the corresponding marginal vectors.

Lemma 5.3.2 *Let $(N, v, E) \in TUC^N$ and let $\sigma \in \Pi(N)$. Then $m^\sigma(v^E) = M^\sigma(V^E)$.*

Proof: Assume without loss of generality that $\sigma(i) = i$ for all $i \in N$. Define $S_i = \{1, \ldots, i\}$ for all $i \in N$. First, $M_1^\sigma(V^E) = 0 = m_1^\sigma(v^E)$. Next, let $k \in N\backslash\{n\}$ and assume that $M_j^\sigma(V^E) = m_j^\sigma(v^E)$ for all $j \in \{1, \ldots, k\}$. Let $T_k \subset S_k$ be such that $T_k \cup \{k+1\} \in S_{k+1}/E$ and define $\bar{T}_k = T_k \cup \{k+1\}$. Then,

$$\begin{aligned}
M_{k+1}^\sigma(V^E) &= \max\{x \,|\, (M_{S_k}^\sigma, x) \in V^E(S_{k+1})\} \\
&= \max\{x \,|\, (M_{T_k}^\sigma, x) \in V^E(\bar{T}_k)\} \\
&= \max\{x \,|\, \sum_{i\in T_k} M_i^\sigma(V^E) + x \leq v(\bar{T}_k)\} \\
&= v(\bar{T}_k) - \sum_{i\in T_k} M_i^\sigma(V^E) \\
&= v^E(T_k \cup \{k+1\}) - \sum_{i\in T_k} m_i^\sigma(v^E) \\
&= m_{k+1}^\sigma(v^E).
\end{aligned}$$

□

The Shapley value of the TU graph-restricted game is called the *Myerson value* of the communication situation (cf. Myerson (1977)). This value $\mu : TUC^N \to \mathbb{R}^N$ is defined as

$$\mu(N, v, E) = \Phi(v^E).$$

Theorem 5.3.3 *Let $(N, v, E) \in TUC^N$. Then $\mu(N, v, E) = MC(V^E)$.*

Proof: It follows from Lemma 5.3.2 that $\mu(N, v, E)$ is a compromise between 0 and $\sum_{\sigma \in \Pi(N)} M^\sigma(V^E)$:

$$\mu(N, v, E) = \beta_V \sum_{\sigma \in \Pi(N)} M^\sigma(V^E),$$

where $\beta_V = \max\{\beta \in \mathbb{R}_+ \,|\, \beta \sum_{\sigma \in \Pi(N)} M^\sigma(V) \in Z^{N, v^E(N)}\}$. By construction, $\sum_{i \in C} \mu_i(N, v, E) = v(C)$ for all $C \in N/E$ (cf. Myerson (1977)). So, $\mu(N, v, E) \in V^E(N) = \prod_{C \in N/E} Z^{C, v(C)}$. Then, from the definition of $MC(V^E)$ and $V^E(N) \subset Z^{N, v^E(N)}$, the assertion follows. □

5.4 Inheritance of properties

In this section we study the inheritance of properties in NTU communication situations. Ie, we provide necessary and sufficient conditions that a network (N, E) must satisfy so that for every game $V \in NTU^N$ that satisfies a certain property, the graph-restricted game V^E satisfies the same property. Most of our results are based on their TU counterparts in Van den Nouweland and Borm (1991) and Slikker (2000) and in many proofs, counterexamples with NTU games arising from TU games are used. It turns out that the necessary and sufficient conditions on the network are the same for TU and NTU games for many properties, with the notable exception of (individual merge) convexity.

First of all, we characterise when balancedness is inherited.

Theorem 5.4.1 *Let (N, E) be a communication network. Then the following two statements are equivalent.*

(i) (N, E) is connected or empty.

(ii) For all balanced $V \in NTU^N$, V^E is balanced.

Proof: "(i)$\Rightarrow$(ii)" Assume that (i) holds. If (N,E) is empty, then for all $V \in NTU^N$, $V^E(S) = \bar{Z}^{S,0}$ for all $S \subset N, S \neq \emptyset$ and hence, V^E is balanced. So, assume that (N,E) is connected and let $V \in NTU^N$ be balanced. Let $x \in C(V)$. Because (N,E) is connected, $x \in V^E(N) = V(N)$. Let $S \subset N, S \neq \emptyset$. Because $x \in C(V)$, there does not exist a $y \in V(C)$ such that $y > x_C$ for any $C \in S/E$. But since $V^E(S) = \prod_{C \in S/E} V(C)$, there does not exist a $y \in V^E(S)$ such that $y > x_S$. Hence, $x \in C(V^E)$ and V^E is balanced.
"(ii)$\Rightarrow$(i)" Assume that (ii) holds. If $|N| \leq 2$, the statement is trivial, so assume that $|N| \geq 3$ and suppose (N,E) is not connected. Take $V(S) = Z^{S,-1}$ for all $S \subset N, 1 < |S| < |N|$ and $V(N) = Z^{N,0}$. Then V is balanced, since $(0,\ldots,0) \in C(V)$. By assumption, V^E is balanced as well, so let $y \in C(V^E)$. Then $y_i \geq V^E(\{i\}) = 0$ for all $i \in N$. We also have $\sum_{i \in N} y_i = \sum_{C \in N/E} \sum_{i \in C} y_i$. Since for each component $C \in N/E$ it holds that $\sum_{i \in C} y_i$ is at most either 0 or -1, it follows from $y \geq 0$ that N/E can only contain components C with $\sum_{i \in C} y_i = 0$. Since (N,E) is not connected, it must be empty. $\square$

Contrary to balancedness, total balancedness is always inherited, as is shown in the following proposition. We denote the subgame of a restricted game V^E with respect to coalition $S \subset N, S \neq \emptyset$ by $V^{E,S}$.

Proposition 5.4.2 *Let $(N,V,E) \in NTUC^N$. If V is totally balanced, then V^E is totally balanced.*

Proof: Assume that V is totally balanced. Let $T \subset N, T \neq \emptyset$. Then there exists an $x^C \in C(V^C)$ for all $C \in T/E$. Define $x \in \mathbb{R}^T$ by $x_C = x^C$ for all $C \in T/E$. It suffices to show that $x \in C(V^{E,T})$.
Since $x^C \in V^C(C) = V^{E,T}(C)$ for all $C \in T/E$, we have $x \in V^{E,T}(T)$. Next, let $S \subset T, S \neq \emptyset$. Suppose there exists a $y \in V^{E,T}(S)$ such that $y > x_S$. Let $D \in S/E$ and let $C \in T/E$ be such that $D \subset C$. Then $y_D > x_D$ and $y_D \in V^{E,T}(D) = V^C(D)$ contradict $x^C \in C(V^C)$. Hence, there exists no $y \in V^{E,T}(S)$ such that $y > x_S$ and so, V^E is totally balanced. $\square$

Superadditivity is also inherited for every communication network.

Proposition 5.4.3 *Let $(N,V,E) \in NTUC^N$. If V is superadditive, then V^E is superadditive.*

Proof: Assume that V is superadditive. Let $S, T \subset N, S \cap T = \emptyset, S, T \neq \emptyset$. Note that $(S/E) \cup (T/E)$ is a finer partition of $S \cup T$ than $(S \cup T)/E$. Now,

$$V^E(S) \times V^E(T) = \prod_{C \in S/E} V(C) \times \prod_{C \in T/E} V(C) \subset \prod_{C \in (S \cup T)/E} V(C) = V^E(S \cup T),$$

where the inclusion follows from combining the components of $(S/E) \cup (T/E)$ and using superadditivity of V. □

In a similar way, one can prove that individual superadditivity is always inherited.

Lemma 5.4.4 *Let $(N, V, E) \in NTUC^N$. If V is individually superadditive, then V^E is individually superadditive.*

Van den Nouweland and Borm (1991) show that TU convexity is inherited for all cycle-complete graphs. The following lemma shows that cycle-completeness is also necessary for individual merge convexity in NTU games (see section 3.2).

Lemma 5.4.5 *Let (N, E) be a network which is not cycle-complete. Then there exists an individual merge convex game $V \in NTU^N$ such that V^E is not individual merge convex.*

Proof: Van den Nouweland and Borm (1991) show that there exists a convex game $v \in TU^N$ such that v^E is not convex. Let $v \in TU^N$ be such a game and let V be the corresponding NTU game. Then V is individual merge convex. Because a TU game is convex if and only if all its marginal vectors belong to the core, there exists a $\sigma \in \Pi(N)$ such that $m^\sigma(v^E) \notin C(v^E)$. But then, because of Proposition 5.3.1 and Lemma 5.3.2, $M^\sigma(V^E) \notin C(V^E)$. Hence, V^E is not marginal convex and therefore not individual merge convex. □

However, cycle-completeness is not sufficient for inheritance of individual merge convexity, as is shown in the following example.

Example 5.4.1 Consider $(N, V, E) \in NTUC^N$ with $N = \{1, 2, 3, 4\}$,

$$\begin{aligned} V(S) &= \bar{Z}^{S,1} \text{ if } S \in \{\{1,3\},\{3,4\},\{2,3,4\}\}, \\ V(\{1,2,3\}) &= cc(\{(1,0,0),(0,2,0),(0,0,1)\}), \\ V(\{1,3,4\}) &= \bar{Z}^{\{1,3,4\},2}, \end{aligned}$$

$$\begin{aligned}
V(N) &= cc(\bar{Z}^{N,2} \cup \{(0,2,0,1)\}), \\
V(S) &= \bar{Z}^{S,0} \text{ for other } S \subset N, S \neq \emptyset
\end{aligned}$$

and $E = \{\{1,2\},\{2,3\},\{3,4\}\}$.

This game is individual merge convex. The graph-restricted game V^E is given by

$$\begin{aligned}
V^E(S) &= \bar{Z}^{S,1} \text{ if } S \in \{\{3,4\},\{2,3,4\}\}, \\
V^E(\{1,2,3\}) &= cc(\{(1,0,0),(0,2,0),(0,0,1)\}), \\
V^E(\{1,3,4\}) &= cc(\{(0,1,0),(0,0,1)\}), \\
V^E(N) &= cc(\bar{Z}^{N,2} \cup \{(0,2,0,1)\}), \\
V^E(S) &= \bar{Z}^{S,0} \text{ for other } S \subset N, S \neq \emptyset.
\end{aligned}$$

The graph-restricted game is not individual merge convex: take $k = 2$, $S = \{1,3\}$ and $T = \{1,3,4\}$, and take $p = (0,0) \in WPar(V^E, S) \cap IR(V^E, S)$, $q = (0,1,0) \in V^E(T)$ and $r = (0,2,0) \in V^E(S \cup \{k\})$. Then there does not exist an $s \in V^E(T \cup \{k\})$ such that $s_2 \geq 2$ and $s_3 \geq 1$. Hence, V^E is not individual merge convex.

For later purposes, note that if we take $E' = E \cup \{\{2,4\}\}$, we get the same graph-restricted game $V^{E'} = V^E$ and hence, $V^{E'}$ is not individual merge convex. $\triangleleft$

It turns out that for NTU games, individual merge convexity is inherited for graphs whose components are either complete or a star.

Theorem 5.4.6 *Let (N, E) be a communication network. Then the following two statements are equivalent.*

(i) For all $C \in N/E$, $(C, E(C))$ is complete or a star.

(ii) For all individual merge convex $V \in NTU^N$, V^E is individual merge convex.

Proof: "(i)⇒(ii)" Assume that (i) holds. Let $V \in NTU^N$ be an individual merge convex game. Then it follows from Lemma 5.4.4 that V^E is individually superadditive. For the individual merge property, let $k \in N$, $S \subsetneq T \subset N \backslash \{k\}, S \neq \emptyset$, and let $p \in WPar(V^E, S) \cap IR(V^E, S)$, $q \in V^E(T)$ and $r \in V^E(S \cup \{k\})$ be such that $r_S \geq p$. Let $C^k \in N/E$ be such that $k \in C^k$ and denote $S^k = S \cap C^k$ and $T^k = T \cap C^k$. Then it suffices to show that $(q_{T^k}, r_k) \in V^E(T^k \cup \{k\})$.

First, suppose C^k is complete. Then $S^k \in S/E$, $T^k \in T/E$ and $S^k \cup \{k\} \in (S \cup \{k\})/E$, so $p_{S^k} \in WPar(V, S^k) \cap IR(V, S^k)$, $q_{T^k} \in V(T^k)$ and $r_{S^k \cup \{k\}} \in V(S^k \cup \{k\})$. Since V is individual merge convex, $(q_{T^k}, r_k) \in V(T^k \cup \{k\}) = V^E(T^k \cup \{k\})$.
Second, suppose that C^k is a star. If k is at the centre of this star, then $(S^k, E(S^k))$ and $(T^k, E(T^k))$ are empty and $(q_{T^k}, r_k) \in V^E(T \cup \{k\})$ by individual superadditivity. If k is not at the centre, but a member of S^k is, then the same argument as in the case where C^k is complete can be used. If neither k nor a member of S^k is at the centre, then $r_k = 0$ and $(q_{T^k}, r_k) \in V^E(T \cup \{k\})$ by individual superadditivity.
"(ii)⇒(i)" Assume that (ii) holds. It follows from Lemma 5.4.5 that (N, E) is cycle-complete. Suppose there exists a component $C \in N/E$ which is not complete or a star. If $(C, E(C))$ does not contain a cycle, then there exist four players in C, without loss of generality $M = \{1, \dots, 4\} \subset C$, such that $E(M) = \{\{1,2\}, \{2,3\}, \{3,4\}\}$. Otherwise, it follows from Lemma 4.2 in Slikker (2000) that there exists, without loss of generality, $M = \{1, \dots, 4\} \subset C$, such that $E(M) = \{\{1,2\}, \{2,3\}, \{3,4\}, \{2,4\}\}$. Take $V \in NTU^N$ such that the subgame (M, V^M) equals the game in Example 5.4.1 and the players in $N \backslash M$ are dummy players. This game is individual merge convex, but the graph-restricted game V^E is not. This contradicts (ii), so every every component must be complete or a star. □

Although cycle-completeness is not sufficient to ensure inheritance of individual merge convexity for arbitrary NTU games, it is sufficient for NTU games arising from TU games.

Proposition 5.4.7 *Let (N, E) be a communication network. Then the following two statements are equivalent.*

(i) (N, E) is cycle-complete.

(ii) For every convex game $v \in TU^N$, V^E is individual merge convex.

Proof: "(i)⇒(ii)" Assume that (i) holds. Let $v \in TU^N$ be a convex game. Then the corresponding NTU game V is individually superadditive and by Lemma 5.4.4, V^E is individually superadditive as well.
For the individual merge property, let $k \in N$, $S \subsetneq T \subset N \backslash \{k\}, S \neq \emptyset$ and let $p \in WPar(V^E, S) \cap IR(V^E, S)$, $q \in V^E(T)$, $r \in V^E(S \cup \{k\})$ be such that $r_S \geq p$. Define $\mathcal{C} = \{C \in S/E \,|\, \exists_{i \in C} : \{i,k\} \in E\}$, $\mathcal{C}' = (S/E) \backslash \mathcal{C}$, $\mathcal{D} = \{D \in T/E \,|\, \exists_{i \in D} : \{i,k\} \in$

$E\}$ and $\mathcal{D}' = (T/E)\backslash\mathcal{D}$. Because $p \in WPar(V^E, S)$, we have that $\sum_{i\in C} p_i = v(C)$ for all $C \in S/E$. Also, $\sum_{i\in D} q_i \leq v(D)$ for all $D \in T/E$, $r_k + \sum_{i\in\bigcup_{C\in\mathcal{C}} C} r_i \leq v(\{k\} \cup \bigcup_{C\in\mathcal{C}} C)$ and $\sum_{i\in C} r_i \leq v(C)$ for all $C \in \mathcal{C}'$. From the proof of Theorem 1 in Van den Nouweland and Borm (1991), we know that

$$v(\{k\} \cup \bigcup_{C\in\mathcal{C}} C) - \sum_{C\in\mathcal{C}} v(C) \leq v(\{k\} \cup \bigcup_{D\in\mathcal{D}} D) - \sum_{D\in\mathcal{D}} v(D).$$

So, subsequently we have

$$\begin{aligned} v(\{k\} \cup \bigcup_{C\in\mathcal{C}} C) + \sum_{C\in\mathcal{C}'} v(C) - \sum_{C\in S/E} v(C) &\leq v(\{k\} \cup \bigcup_{D\in\mathcal{D}} D) - \sum_{D\in\mathcal{D}} v(D), \\ r_k + \sum_{i\in\bigcup_{C\in\mathcal{C}} C} r_i + \sum_{i\in\bigcup_{C\in\mathcal{C}'} C} r_i - \sum_{i\in S} p_i &\leq v(\{k\} \cup \bigcup_{D\in\mathcal{D}} D) - \sum_{i\in\bigcup_{D\in\mathcal{D}} D} q_i, \\ v(\{k\} \cup \bigcup_{D\in\mathcal{D}} D) &\geq r_k + \sum_{i\in\bigcup_{D\in\mathcal{D}} D} q_i, \end{aligned}$$

where in the last step we use $r_S \geq p$. Then, because

$$V^E(T \cup \{k\}) = Z^{\bigcup_{D\in\mathcal{D}} D\cup\{k\}, v(\bigcup_{D\in\mathcal{D}} D\cup\{k\})} \times \prod_{D\in\mathcal{D}'} Z^{D,v(D)},$$

we have that $(q, r_k) \in V^E(T \cup \{k\})$ and hence, V^E is individual merge convex.
"(ii)⇒(i)" Follows from the proof of Lemma 5.4.5. □

As is the case for TU games, existence of a population monotonic allocation scheme, or pmas (see section 4.2), is always inherited for NTU games, as is shown in the following proposition.

Proposition 5.4.8 *Let $(N, V, E) \in NTUC^N$. If V has a pmas, then V^E has a pmas.*

Proof: Assume that V has a pmas $(y^S)_{S\subset N, S\neq\emptyset}$. For $i \in S$, denote by $C_i(S)$ the component in S/E to which i belongs. Define $x_i^S = y_i^{C_i(S)}$ for all $S \subset N, i \in S$. Because $y^C \in WPar(V, C)$ for all $C \in S/E$, we have that $x^S \in WPar(V^E, S)$. Furthermore, for all $i \in S \subset T \subset N$,

$$x_i^S = y_i^{C_i(S)} \leq y_i^{C_i(T)} = x_i^T,$$

because $C_i(S) \subset C_i(T)$ and $(y^S)_{S\subset N, S\neq\emptyset}$ is a pmas. Hence, $(x^S)_{S\subset N, S\neq\emptyset}$ is a pmas for the game V^E. □

Now we turn our attention to the MC value (see section 3.6.1). One interesting question is whether the MC value is an element of the core. The following proposition shows the relationship between an NTU game and its graph-restricted game in terms of this question.

Proposition 5.4.9 *Let (N, E) be a communication network. Then the following two statements are equivalent.*

(i) (N, E) is complete or empty.

(ii) For every $V \in NTU^N$ with $MC(V) \in C(V)$, $MC(V^E) \in C(V^E)$.

Proof: "(i)⇒(ii)" Trivial.
"(ii)⇒(i)" Assume that (ii) holds. Suppose (N, E) is not complete or empty. Then, along the lines of Theorem 4.1 in Slikker (2000), for the game V described in the proof of Theorem 5.4.1, we have that $MC(V) \in C(V)$, but $MC(V^E) \notin C(V^E)$. This contradicts (ii), so (i) must hold. □

Another interesting question is whether the *MC allocation scheme* $(MC(V^S))_{S \subset N, S \neq \emptyset}$ is a pmas for the NTU game V. To characterise when this property is inherited, we need the following lemma.

Lemma 5.4.10 *Let $V \in NTU^N$ and let $i \in N$ be a dummy player. Then $MC_i(V) = 0$ and $MC_{N \backslash \{i\}}(V) = MC(V^{N \backslash \{i\}})$.*

Proof: By construction, $M_i^\sigma(V) = 0$ for all $\sigma \in \Pi(N)$ and hence, $MC_i(V) = \alpha_V \sum_{\sigma \in \Pi(N)} M_i^\sigma(V) = 0$. Furthermore, $M_j^\sigma(V) = M_j^{\sigma|N \backslash \{i\}}(V^{N \backslash \{i\}})$ for all $j \in N \backslash \{i\}, \sigma \in \Pi(N)$ and $V(N \backslash \{i\}) = V^{N \backslash \{i\}}(N \backslash \{i\})$, so $MC_{N \backslash \{i\}}(V) = MC(V^{N \backslash \{i\}})$.
□

For TU games, the property that the Shapley allocation scheme is a pmas is inherited for graphs whose components are all complete. This is also necessary for NTU games, as is illustrated in the following example.

Example 5.4.2 Consider $(N, V, E) \in NTUC^N$ with $N = \{1, 2, 3\}$, $E = \{\{1, 2\}, \{2, 3\}\}$, $V(S) = Z^{S,2}$ for $S \subset N, |S| = 2$ and $V(N) = Z^{N,3}$. It is readily checked that the MC allocation scheme $(MC(V^S))_{S \subset N, S \neq \emptyset}$ is a pmas for V. However, in the graph-restricted game V^E,

$$MC_1(V^{E,\{1,2\}}) = 1 > \frac{2}{3} = MC_1(V^E).$$

◁

However, completeness of every component is not sufficient, as is shown in the following example.

Example 5.4.3 Consider $(N, V, E) \in NTUC^N$ with $N = \{1, \ldots, 4\}$, $E = \{\{1, 2\}, \{3, 4\}\}$ and

$$\begin{aligned} V(\{1,2\}) &= Z^{\{1,2\},1}, \\ V(\{3,4\}) &= \{x \in \mathbb{R}^{\{3,4\}} \mid x_3 \leq 1, x_4 \leq 1\}, \\ V(S \cup \{i\}) &= V(S) \times V(\{i\}) \text{ for } S \in \{\{1,2\},\{3,4\}\}, i \in N \backslash S, \\ V(N) &= Z^{N,4}, \\ V(S) &= \bar{Z}^{S,0} \text{ for other } S \subset N, S \neq \emptyset. \end{aligned}$$

It is readily checked that $(MC(V^S))_{S \subset N, S \neq \emptyset}$ is a pmas for (N, V). However, in the graph-restricted game (N, V^E),

$$MC_3(V^{E,\{3,4\}}) = 1 > \frac{1}{2} = MC_3(V^E),$$

where $\alpha_{V^{E,\{3,4\}}} = 1$ and $\alpha_{V^E} = \frac{1}{24}$. Note that the MC value is not component decomposable: in the grand coalition, players 3 and 4 suffer from the presence of players 1 and 2 in the sense that the maximum α is restricted by coalition $\{1, 2\}$.

Moreover, because the MC allocation scheme is a pmas, $MC(V^S) \in C(V^S)$ for all $S \subset N, S \neq \emptyset$. However, the MC value of the graph-restricted game is not even a core element:

$$MC(V^E) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \notin C(V^E).$$

◁

Proposition 5.4.11 *Let (N, E) be a communication network. Then the following two statements are equivalent.*

(i) If $C \in N/E, |C| > 1$, then $(C, E(C))$ is complete and $|D| = 1$ for all $D \in N/E, D \neq C$.

(ii) For all $V \in NTU^N$ such that $(MC(V^S))_{S\subset N,S\neq\emptyset}$ is a pmas for V, $(MC(V^{E,S}))_{S\subset N,S\neq\emptyset}$ is a pmas for V^E.

Proof: "(i)$\Rightarrow$(ii)" Assume that (i) holds. If $|C| = 1$ for all $C \in N/E$, then (ii) holds trivially. So, let $C \in N/E, |C| > 1$, and let $V \in NTU^N$ be such that the MC allocation scheme is a pmas. By Lemma 5.4.10, $MC_i(V^{E,S}) = 0$ for all $S \subset N, i \in S\backslash C$ and $MC_C(V^E) = MC(V^{E,C})$. Because C is complete, $MC(V^{E,S}) = MC(V^S)$ for all $S \subset C, S \neq \emptyset$. So, since $(MC(V^S))_{S\subset C,S\neq\emptyset}$ is a pmas for (C, V^C), it is also a pmas for $(C, V^{E,C})$ and (ii) holds.
"(ii)$\Rightarrow$(i)" Assume that (ii) holds. Suppose that there is a component $C \in N/E$ such that $(C, E(C))$ is not complete. Then there exists, without loss of generality, $M = \{1, 2, 3\} \subset C$ such that $E(M) = \{\{1, 2\}, \{2, 3\}\}$. Take $V \in NTU^N$ such that (M, V^M) is the game in Example 5.4.2 and the players in $N\backslash M$ are dummy players. Then, as a result of Lemma 5.4.10, $(MC(V^{E,S}))_{S\subset N,S\neq\emptyset}$ is not a pmas for V^E, although $(MC(V^S))_{S\subset N,S\neq\emptyset}$ is a pmas for V. Contradiction, so there is no incomplete component.
Next, suppose there exist two complete components with more than one player. Then, without loss of generality, there exists $M = \{1, \ldots, 4\} \subset N$ such that $E(M) = \{\{1, 2\}, \{3, 4\}\}$. Take $V \in NTU^N$ such that (M, V^M) is the game in Example 5.4.3 and the players in $N\backslash M$ are dummy players. Then $(MC(V^S))_{S\subset N,S\neq\emptyset}$ is a pmas for V, but $(MC(V^{E,S}))_{S\subset N,S\neq\emptyset}$ is not a pmas for V^E. This contradicts (ii), so (i) must hold. $\square$

Finally, we consider the MC values of all subgames. Slikker (2000) shows that for TU games, the property that the Shapley value of each subgame lies in its corresponding core is inherited for graphs with complete or star components. In the following example, we show that for NTU games, completeness of each component is necessary.

Example 5.4.4 Consider $(N, V, E) \in NTUC^N$ with $N = \{1, 2, 3\}$, $E = \{\{1, 2\}, \{2, 3\}\}$ and

$$\begin{aligned} V(\{1,2\}) &= \{x \in \mathbb{R}^{\{1,2\}} \,|\, x_1 \leq 1, x_2 \leq 2\}, \\ V(\{1,3\}) &= \{x \in \mathbb{R}^{\{1,3\}} \,|\, x_1 \leq 2, x_3 \leq 0\}, \\ V(\{2,3\}) &= Z^{\{2,3\},1}, \\ V(N) &= Z^{N,3}. \end{aligned}$$

It is readily checked that $MC(V^S) \in C(V^S)$ for all $S \subset N, S \neq \emptyset$. However, in the graph-restricted game V^E,

$$MC(V^E) = (\frac{5}{6}, \frac{9}{6}, \frac{4}{6}),$$

which is not a core element because of coalition $\{1, 2\}$. $\triangleleft$

Furthermore, we need the following lemma.

Lemma 5.4.12 *Let $(N, V, E) \in NTUC^N$ and let $i \in N$ be a dummy player. Then*

$$C(V) = \{(x, 0) \in \mathbb{R}^N \mid x \in C(V^{N\setminus\{i\}})\}.$$

Using this, we can state our final result of this chapter.

Proposition 5.4.13 *Let (N, E) be a communication network. Then the following two statements are equivalent.*

(i) If $C \in N/E, |C| > 1$, then $(C, E(C))$ is complete and $|D| = 1$ for all $D \in N/E, D \neq C$.

(ii) For all $V \in NTU^N$ such that $MC(V^S) \in C(V^S)$ for all $S \subset N, S \neq \emptyset$, $MC(V^{E,S}) \in C(V^{E,S})$ for all $S \subset N, S \neq \emptyset$.

Proof: "(i)⇒(ii)" Follows immediately from Lemmas 5.4.10 and 5.4.12.
"(ii)⇒(i)" Assume that (ii) holds. Suppose that there exists a $C \in N/E$ which is not complete. Then the game in Example 5.4.4 can be used to contradict (ii). If there is more than one component with more than one player, Example 5.4.3 can be used. Hence, (i) must hold. $\square$

To round of this chapter, we summarise our inheritance results in the following table, together with the corresponding results for TU games from Van den Nouweland and Borm (1991) and Slikker (2000).

Property	*TU inheritance condition*	*NTU inheritance condition*
Balancedness	connected or empty	connected or empty
Total balancedness	no condition	no condition
Superadditivity	no condition	no condition
Individual merge convexity	cycle-complete	every component complete or star
Existence of pmas	no condition	no condition
MC value in core	complete or empty	complete or empty
MC allocation scheme pmas	every component complete	one component complete, others singletons
MC value of every subgame in core	every component complete or star	one component complete, others singletons

Chapter 6

Spillover games

6.1 Introduction

In standard noncooperative game theory it is assumed that players cannot make binding agreements. That is, each cooperative outcome must be sustained by Nash equilibrium strategies. At the other end of the spectrum, in cooperative game theory, players have no choice but to cooperate. The standard transferable utility (TU) model assumes that all players involved want to come to an agreement and the main task is to propose socially acceptable solutions. Noncooperative theory tries to predict the outcome of strategic situations using equilibrium concepts that at least require the predicted strategy combinations to be robust against unilateral deviations.

Both approaches seem to be diametrically opposed. Many real life situations, however, exhibit both cooperative and strategic features. Neither approach suffices in these cases. Examples of these situations can be found in parliaments where governments are based on multiple-party coalitions. Here noncooperative theory obviously does not work, since agreements have to be made. Also TU theory is not sufficiently rich, since typically not all parties represented in parliament are part of the government. Furthermore, TU theory does not take into account the spillover effects from coalitions on the parties outside. These spillovers measure the impact of government policy on the opposition parties and thus reflect in some way the parties' relative positions in the political spectrum.

This kind of spillovers is present in many situations. For example, one can think of a situation where a group of people needs to be connected to a source, like in a telecommunication network. In the literature, many solution concepts have been introduced (cf. Bird (1976) and Feltkamp (1995)), but these do not take into account the strategic considerations of players not to join the public enterprise. However, spillovers occur if one assumes that publicly accessible networks can be built by smaller groups, hence creating a special type of free-rider effect. In order to find a fair solution, these possible spillovers should be taken into account.

From these examples one can conclude that in many cooperative situations, a socially acceptable solution concept should incorporate the strategic options that result from spillovers. Essentially, spillovers induce a noncooperative aspect in cooperative situations. They provide incentives for players to join or to stay out of a coalition. In TU games, these spillovers are not taken into account, but one implicitly assumes that players do not have a better alternative than to stay in the group. As mentioned before, in the government example, this is typically not the case.

To capture spillovers in a cooperative model, this chapter, which is based on Thijssen et al. (2002), introduces a new class of games, namely *spillover games*. This class of games builds on ideas introduced in Van der Rijt (2000) for government situations. In a spillover game, each coalition is assigned a value, as in a TU game. In addition, all the players outside the coalition are separately assigned a value as well, capturing the spillovers from the coalition to the outside players. We restrict ourselves to a coalitional structure where there is one coalition (eg, a government or a group building a public network) and a group of singletons outside. This allows us to redefine some basic concepts of TU theory, while not assuming *ex ante* that all players are fully cooperating.

The model of spillover games is explicitly aimed at analysing the influence of a coalition S on the payoffs of the players outside S. In this sense, spillover games differ fundamentally from games in partition function form (cf. Bloch (1996) and Yi (1997)), where for each coalition S the influence of the possible coalition structures on the player set outside S on the payoff to coalition S is analysed. Hence, the causality of spillovers in spillover games is reversed compared to partition function form games.

The structure of this chapter is as follows. In section 2 the class of spillover games is introduced. In section 3 we extend various TU notions (eg, core, nucleolus, convexity) to this new class of games and generalise some basic results. In section 4 we use a government formation example to introduce marginal vectors and a Shapley value. Section 5 takes a closer look at public-private connection problems and discusses some other classes of OR games where our spillover model seems appropriate.

6.2 The model

A *spillover game* is a tuple $\mathcal{G} = (N, \mathcal{W}, v, z)$, where $N = \{1, \ldots, n\}$ is the set of players, $\mathcal{W} \subset 2^N$ is a set of coalitions that can cooperate and v and z are payoff functions, to be specified below.

One main feature of our model is the assumption that exactly one coalition of players will cooperate. Contrary to TU games, however, we do not impose that the resulting coalition is the grand coalition. In the example of government formation, the grand coalition would be a very extreme outcome.

The set $\mathcal{W} \subset 2^N$ contains those coalitions of players who can actually cooperate. An element of $\mathcal{W}$ is called a *winning coalition*. In a government situation, a natural choice for $\mathcal{W}$ is the collection of coalitions that have a majority in parliament.

We assume that $\mathcal{W}$ satisfies the following properties:

- $N \in \mathcal{W}$.
- $S \subset T, S \in \mathcal{W} \Rightarrow T \in \mathcal{W}$ (monotonicity).

The first property ensures that the game is not trivial, in the sense that there is at least one winning coalition. The second property states that if a small group of players S can cooperate (eg, have a majority), then a larger coalition $T \supset S$ is also winning.

The (nonnegative) payoff function $v : 2^N \to \mathbb{R}_+$ assigns to every coalition $S \subset N$ a value $v(S)$. If $S \in \mathcal{W}$, then $v(S)$ represents the total payoff to the members of S in case they cooperate. For $S \notin \mathcal{W}$ we simply impose $v(S) = 0$.

Suppose that the players in S cooperate. Then the members of S do not only generate a payoff to themselves. Their cooperation also affects the players outside S. The payoffs to the other players, which are called *spillovers* (with respect to S), are given by the vector $z^S \in \mathbb{R}_+^{N \setminus S}$. Again, we simply put $z^S = 0$ for $S \notin \mathcal{W}$. Note that whereas the members of S still have the freedom to divide the amount $v(S)$ among themselves, the payoffs to the players outside S are individually fixed.

Spillovers (with respect to S) are called *positive* if the total payoff to every coalition $U \subset N \setminus S$ is higher than what U can earn on its own, so if

$$\sum_{i \in U} z_i^S \geq v(U) \tag{6.1}$$

for every $U \subset N \backslash S$. Likewise, spillovers are *negative* if for every $U \subset N \backslash S$ the reverse inequality holds in (6.1). Note that if for different coalitions U not the same inequality holds, spillovers are neither positive nor negative.

A set of winning coalitions $\mathcal{W} \subset 2^N$ is called *N-proper* (cf. Feltkamp (1995)) if $S \in \mathcal{W}$ implies $N \backslash S \notin \mathcal{W}$. In the context of coalition formation in politics, this property relates to the fact that a coalition and its complement can not have a majority at the same time.

6.3 Some basic results

In this section, we define and analyse some rules for and properties of spillover games. These notions are based on their well-known analogues for TU games.

A payoff vector $x \in \mathbb{R}^N$ is individually rational if $x_i \geq v(\{i\})$ for all $i \in N$. The *S-imputation set* of a spillover game $\mathcal{G} = (N, \mathcal{W}, v, z)$ for $S \in \mathcal{W}$, $I_S(\mathcal{G})$, consists of those individually rational payoff vectors in $\mathbb{R}_+^N$ which allocate $v(S)$ among the members of S, while giving the members of $N \backslash S$ their spillovers, ie,

$$I_S(\mathcal{G}) = \{x \in \mathbb{R}^N \mid \sum_{i \in S} x_i = v(S), x_{N \backslash S} = z^S, \forall_{i \in N} : x_i \geq v(\{i\})\}.$$

The *imputation set* of $\mathcal{G}$ is defined by

$$I(\mathcal{G}) = \bigcup_{S \in \mathcal{W}} I_S(\mathcal{G}).$$

It follows from individual rationality that every imputation vector is nonnegative.

A payoff vector in the S-imputation set belongs to the S-core if for every coalition, the total payoff to the members of that coalition exceeds its value. So, for $S \in \mathcal{W}$ we define the S-core by

$$C_S(\mathcal{G}) = \{x \in \mathbb{R}^N \mid \sum_{i \in S} x_i = v(S), x_{N \backslash S} = z^S, \forall_{T \subset N} : \sum_{i \in T} x_i \geq v(T)\},$$

or equivalently,

$$C_S(\mathcal{G}) = \{x \in \mathbb{R}_+^N \mid \sum_{i \in S} x_i = v(S), x_{N \backslash S} = z^S, \forall_{T \in \mathcal{W}} : \sum_{i \in T} x_i \geq v(T)\}.$$

An element of the S-core is stable in the sense that there is no other winning coalition T that objects to the proposed allocation on the basis of it being able to obtain more if it forms. The *core* of $\mathcal{G}$ consists of all Pareto efficient payoff vectors in the union of all S-cores, so

$$C(\mathcal{G}) = Par(\bigcup_{S \in \mathcal{W}} C_S(\mathcal{G})),$$

where $Par(A) = \{x \in A \mid \neg\exists_{y \in A} : y \gneqq x\}$. It follows immediately from the definitions that $C_S(\mathcal{G}) \subset I_S(\mathcal{G})$ for all $S \in \mathcal{W}$ and $C(\mathcal{G}) \subset I(\mathcal{G})$.

Example 6.3.1 Consider the spillover game $\mathcal{G} = (N, \mathcal{W}, v, z)$ with $N = \{1, 2, 3\}$, $\mathcal{W} = \{\{1\}, \{1, 2\}, \{1, 3\}, N\}$ and the following payoffs:

S	$\{1\}$	$\{1,2\}$	$\{1,3\}$	N
$v(S)$	1	5	2	6
z^S	$(1,1)$	3	5	

Then

$$\begin{aligned}
C_{\{1\}}(\mathcal{G}) &= \emptyset, \\
C_{\{1,2\}}(\mathcal{G}) &= Conv(\{(1,4,3),(5,0,3)\}), \\
C_{\{1,3\}}(\mathcal{G}) &= Conv(\{(1,5,1),(2,5,0)\}), \\
C_N(\mathcal{G}) &= Conv(\{(6,0,0),(5,0,1),(1,4,1),(2,4,0)\}),
\end{aligned}$$

where $Conv(A) = \{x \in \mathbb{R}^N \mid \exists_{t \in \mathbb{N}} \exists_{x^1,\ldots,x^t \in A} \exists_{(\lambda_1,\ldots,\lambda_t) \in \Delta^t} : x = \sum_{i=1}^t \lambda_i x^i\}$ is the convex hull of $A \subset \mathbb{R}^N$.

Because the sum of the value and the spillovers is highest for $\{1, 2\}$, the core elements corresponding to this coalition cannot be dominated, so $C_{\{1,2\}}(\mathcal{G}) \subset C(\mathcal{G})$. Also, we have $C_{\{1,3\}}(\mathcal{G}) \subset C(\mathcal{G})$, since the payoff to player 2 is strictly higher than in the core elements corresponding to the other winning coalitions. The N-core, however, does not fully belong to the core of the game. Eg, $(1, 4, 1)$ is dominated by $(1, 4, 3) \in C_{\{1,2\}}(\mathcal{G})$ and $(2, 4, 0)$ by $(2, 5, 0) \in C_{\{1,3\}}(\mathcal{G})$. The core of this game is as follows:

$$\begin{aligned}
C(\mathcal{G}) = \;& Conv(\{(1,4,3),(5,0,3)\}) \cup Conv(\{(1,5,1),(2,5,0)\}) \cup \\
& Conv(\{(6,0,0),(5,0,1),(1,4,1),(2,4,0)\}) \backslash \Big(Conv(\{(1,4,1), \\
& (5,0,1)\}) \cup Conv(\{(1,4,1),(2,4,0)\})\Big).
\end{aligned}$$

$\triangleleft$

For TU games, Bondareva (1963) and Shapley (1967) characterised nonemptiness of the core by means of the concept of *balancedness*. We establish a similar result for the class of $\mathcal{W}$-stable spillover games. A game $\mathcal{G} = (N, \mathcal{W}, v, z)$ is called $\mathcal{W}$-*stable* if

$$S, T \in \mathcal{W}, S \cap T = \emptyset \Rightarrow \begin{cases} \sum_{i \in T} z_i^S \geq v(T), \\ \sum_{i \in S} z_i^T \geq v(S). \end{cases}$$

The idea behind $\mathcal{W}$-stability is that there can exist no two disjoint winning coalitions with positive mutual spillovers. For, if two such coalitions are present, the game would have no stable outcome in the sense that both these coalitions would want to form. Note that positive spillover games and spillover games with N-proper $\mathcal{W}$ belong to the class of $\mathcal{W}$-stable games.

A mapping $\lambda : \mathcal{W} \to \mathbb{R}_+$ is called S-*subbalanced* if

$$\sum_{T \in \mathcal{W}} \lambda(T) e_S^T \leq e_S^S.$$

We denote the set of all S-subbalanced mappings by $\mathcal{B}^S$. A game $\mathcal{G} = (N, \mathcal{W}, v, z)$ is S-*subbalanced* if for all $\lambda \in \mathcal{B}^S$ it holds that

$$\sum_{T \in \mathcal{W}} \lambda(T) \left[v(T) - \sum_{i \in (N \setminus S) \cap T} z_i^S \right] \leq v(S).$$

Suppose a winning coalition S forms, giving its members a total payoff of $v(S)$. Next, consider a winning coalition T and consider the situation where T forms. The payoff to T would then be $v(T)$, but some of its members would have to forego the spillovers resulting from the formation of S. So, after subtracting these opportunity costs, the net payoff to T equals the expression inside the brackets. A game is S-subbalanced if dividing the net payoffs of all winning coalitions T in an S-subbalanced way yields a lower payoff than $v(S)$.

Theorem 6.3.1 *Let $\mathcal{G} = (N, \mathcal{W}, v, z)$ be a $\mathcal{W}$-stable spillover game. Then $C(\mathcal{G}) \neq \emptyset$ if and only if there exists an $S \in \mathcal{W}$ such that $\mathcal{G}$ is S-subbalanced.*

Proof: Let $S \in \mathcal{W}$. Then

$$\begin{aligned} & C_S(\mathcal{G}) \neq \emptyset \\ \Leftrightarrow \quad & \{x \in \mathbb{R}_+^N \mid \sum_{i \in S} x_i = v(S), x_{N \setminus S} = z^S, \forall_{T \in \mathcal{W}} : \sum_{i \in T} x_i \geq v(T)\} \neq \emptyset \end{aligned}$$

$$
\begin{aligned}
&\Leftrightarrow \quad v(S) = \min_{x \in \mathbb{R}^N} \{ \sum_{i \in S} x_i \,|\, \forall_{T \in \mathcal{W}} : \sum_{i \in T} x_i \geq v(T), \forall_{i \in S} : x_i \geq 0, x_{N \setminus S} = z^S \} \\
&\overset{(*)}{\Leftrightarrow} \quad v(S) = \max_{\lambda,\mu,\psi} \{ \sum_{T \in \mathcal{W}} \lambda(T) v(T) + \sum_{i \in N \setminus S} \mu_i z_i^S - \sum_{i \in N \setminus S} \psi_i z_i^S \,| \\
&\qquad \sum_{i \in S} \mu_i e^{\{i\}} + \sum_{i \in N \setminus S} \mu_i e^{\{i\}} - \sum_{i \in N \setminus S} \psi_i e^{\{i\}} + \sum_{T \in \mathcal{W}} \lambda(T) e^T = e^S, \lambda, \mu, \psi \geq 0 \} \\
&\Leftrightarrow \quad v(S) = \max_{\lambda,\mu,\zeta} \{ \sum_{T \in \mathcal{W}} \lambda(T) v(T) + \sum_{i \in N \setminus S} \zeta_i z_i^S \,| \\
&\qquad \sum_{i \in S} \mu_i e^{\{i\}} + \sum_{i \in N \setminus S} \zeta_i e^{\{i\}} + \sum_{T \in \mathcal{W}} \lambda(T) e^T = e^S, \lambda, \mu \geq 0 \} \\
&\Leftrightarrow \quad \forall_{\lambda \in \mathcal{B}^S} : \sum_{T \in \mathcal{W}} \lambda(T) v(T) + \sum_{i \in N \setminus S} \zeta_i z_i^S \leq v(S), \\
&\qquad \text{where } \forall_{i \in N \setminus S} : \zeta_i = \sum_{T \in \mathcal{W} : i \in T} \lambda(T) \\
&\Leftrightarrow \quad \forall_{\lambda \in \mathcal{B}^S} : \sum_{T \in \mathcal{W}} \lambda(T) v(T) + \sum_{i \in N \setminus S} \sum_{T \in \mathcal{W} : i \in T} \lambda(T) z_i^S \leq v(S) \\
&\Leftrightarrow \quad \forall_{\lambda \in \mathcal{B}^S} : \sum_{T \in \mathcal{W}} \lambda(T) \left[v(T) - \sum_{i \in (N \setminus S) \cap T} z_i^S \right] \leq v(S).
\end{aligned}
$$

The equivalence (*) follows from duality theory. Note that nonemptiness of the primal feasible set follows from $\mathcal{W}$-stability and that the dual feasible set is always nonempty. Since $C(\mathcal{G}) \neq \emptyset$ if and only if there exists an $S \in \mathcal{W}$ such that $C_S(\mathcal{G}) \neq \emptyset$, the assertion follows. □

A well-known rule for the class of TU games with nonempty imputation set is the nucleolus (see section 2.2). The nucleolus has the appealing property that it lies in the core whenever the core is nonempty (cf. Schmeidler (1969)). For spillover games, we establish a similar result.

The *excess* of coalition $T \subset N$ for imputation $x \in I(\mathcal{G})$ is defined by

$$E(T, x) = v(T) - \sum_{i \in T} x_i.$$

If x is proposed as allocation (corresponding to some winning coalition), the excess of T measures to which extent T is satisfied with x: the lower the excess, the more pleased T is with the proposed allocation. The idea behind the nucleolus is to minimise the highest excesses in a hierarchical manner.

Let $\mathcal{G} = (N, \mathcal{W}, v, z)$ be a game with nonempty imputation set and let $S \in \mathcal{W}$

be such that $I_S(\mathcal{G}) \neq \emptyset$. We define the S-*nucleolus* of $\mathcal{G}$, $\nu_S(\mathcal{G})$ to be the set of S-imputations for which the excesses are lexicographically minimal, ie,

$$\nu_S(\mathcal{G}) = \{x \in I_S(\mathcal{G}) \,|\, \forall_{y \in I_S(\mathcal{G})} : (E(T,x))_{T \subset N} \leq^*_L (E(T,y))_{T \subset N}\}.$$

Because $I_S(\mathcal{G})$ is compact and convex, there exists a unique lexicographic minimum. The *nucleolus* of $\mathcal{G}$, $\nu(\mathcal{G})$, is the lexicographic minimum of all (well-defined) S-nucleoli, which is equivalent to

$$\nu(\mathcal{G}) = \{x \in I(\mathcal{G}) \,|\, \forall_{y \in I(\mathcal{G})} : (E(T,x))_{T \subset N} \leq^*_L (E(T,y))_{T \subset N}\}.$$

Because the imputation set is the finite union of all S-imputation sets, the lexicographic minimum exists. It need not be unique, however, since $I(\mathcal{G})$ itself need not be convex.

As stated before, the TU nucleolus always belongs to the core whenever this set is nonempty. The same holds for spillover games, as is shown in the following proposition.

Proposition 6.3.2 *Let $\mathcal{G} = (N, \mathcal{W}, v, z)$ be such that $C(\mathcal{G}) \neq \emptyset$. Then $\nu(\mathcal{G}) \subset C(\mathcal{G})$.*

Proof: First note that since $C(\mathcal{G}) \neq \emptyset$, the game has a nonempty imputation set and $\nu(\mathcal{G})$ is well-defined. Let $y \in \nu(\mathcal{G})$ and let $S_y \in \mathcal{W}$ be such that $\nu_{S_y}(\mathcal{G}) = \{y\}$. Let $x \in C(\mathcal{G})$. Then the inequalities in the core definition imply $E(T,x) \leq 0$ for all $T \subset N$. But then we must also have $E(T,y) \leq 0$ for all $T \subset N$ and hence, $y \in C_{S_y}(\mathcal{G})$.
To show that y is Pareto efficient, suppose there exists a $z \in \bigcup_{S \in \mathcal{W}} C_S(\mathcal{G})$ such that $z \gneqq y$. Then $E(T,z) \lneqq E(T,y)$ for all $T \subset N$. This contradicts $(E(T,z))_{T \subset N} \geq^*_L (E(T,y))_{T \subset N}$ and hence, $y \in C(\mathcal{G})$. □

A spillover game $\mathcal{G} = (N, \mathcal{W}, v, z)$ is called *superadditive* if

$$v(T) \geq v(S) + \sum_{i \in T \setminus S} z_i^S$$

for all $S \subset T \subset N$. If a game is superadditive, then it is beneficial to form a large coalition: the payoff to S and the individual players in $T \setminus S$ is larger if these two groups merge into one coalition rather than stay separate. Note that if spillovers are positive, this condition is stronger than the TU definition $v(T) \geq v(S) + v(T \setminus S)$, ie, if spillovers are positive, the coalitions have a bigger incentive not to merge.

A weaker version of superadditivity is *individual superadditivity*:

$$v(T) \geq v(\{i\}) + \sum_{j \in T \setminus \{i\}} z_j^{\{i\}}$$

for all $T \subset N, i \in T$. As we did in Chapter 3 for NTU games, we extend the concept of TU convexity to the class of spillover games in various ways.

$\mathcal{G}$ is *convex* if

$$v(S \cup T) + \sum_{i \in S \cap T} z_i^{S \setminus T} \geq v(S) + \sum_{i \in T} z_i^{S \setminus T}$$

for all $S, T \subset N$.

The game $\mathcal{G}$ is *coalition merge convex* if

$$v(T \cup U) - \sum_{j \in T} z_j^U \geq v(S \cup U) - \sum_{j \in S} z_j^U$$

for all $U \subset N, S \subset T \subset N \setminus U$. Coalition merge convexity can be interpreted in terms of increasing marginal contributions: if a large coalition T decides to join U, then its marginal contribution, being the value of the resulting coalition minus the opportunity costs of staying separate, is larger than the marginal contribution (to U) of a smaller coalition S.

Similarly, $\mathcal{G}$ is *individual merge convex* if

$$v(T \cup \{k\}) - \sum_{i \in T} z_i^{\{k\}} \geq v(S \cup \{k\}) - \sum_{i \in S} z_i^{\{k\}}$$

for all $k \in N, S \subset T \subset N \setminus \{k\}$.

Like their TU analogues, convexity and coalition merge convexity turn out to be equivalent, as is shown in the following proposition.

Proposition 6.3.3 *Let $\mathcal{G} = (N, \mathcal{W}, v, z)$ be a spillover game. Then $\mathcal{G}$ is convex if and only if $\mathcal{G}$ is coalition merge convex.*

Proof: "⇒" Let $U \subset N, S \subset T \subset N \setminus U$. Take $A = S \cup U$ and $B = T$. Then,

$$\begin{aligned} v(T \cup U) &= v(A \cup B) \\ &\geq v(A) + \sum_{i \in B} z_i^{A \setminus B} - \sum_{i \in A \cap B} z_i^{A \setminus B} \\ &= v(S \cup U) + \sum_{i \in T} z_i^U - \sum_{i \in S} z_i^U. \end{aligned}$$

"$\Leftarrow$" Let $A, B \subset N$. Take $S = A \cap B$, $T = B$ and $U = A \backslash B$. Then,

$$\begin{aligned} v(A \cup B) &= v(T \cup U) \\ &\geq v(S \cup U) - \sum_{i \in S} z_i^U + \sum_{i \in T} z_i^U \\ &= v(A) - \sum_{i \in A \cap B} z_i^{A \backslash B} + \sum_{i \in B} z_i^{A \backslash B}. \end{aligned}$$

□

As in TU games, (coalition merge) convexity is stronger than superadditivity.

Proposition 6.3.4 *Let $\mathcal{G} = (N, \mathcal{W}, v, z)$ be a convex game. Then $\mathcal{G}$ is superadditive.*

Proof: Let $A \subset B$. Then,

$$\begin{aligned} v(B) &= v(A \cup (B \backslash A)) \\ &\geq v(A) + \sum_{i \in B \backslash A} z_i^{A \backslash (B \backslash A)} - \sum_{i \in A \cap (B \backslash A)} z_i^{A \backslash (B \backslash A)} \\ &= v(A) + \sum_{i \in B \backslash A} z_i^A. \end{aligned}$$

□

In a similar way, individual merge convexity implies individual superadditivity.

It follows immediately from the definitions that every coalition merge convex game is individual merge convex. The reverse is not true, as is shown in the following example.

Example 6.3.2 Consider the spillover game of Example 6.3.1. This game is individual merge convex. However, it is not superadditive: take $S = \{1, 2\}$ and $T = \{1, 2, 3\}$, then $v(T) = 6 < 5 + 3 = v(S) + z_3^S$. ◁

6.4 A Shapley value

In this section, we introduce a Shapley value that can be used to provide an indication of the relative power of the players in a game. On the basis of a government example we introduce the concept of marginal vector for spillover games. Contrary to its TU counterpart, strategic considerations play an important role in our definition of marginal vector.

Example 6.4.1 Consider a parliament with four parties[1]: the communists (COM), socialists (SOC), Christian democrats (CD) and liberals (LIB). The seats are divided as follows:

party	COM	SOC	CD	LIB
share of seats	0.1	0.3	0.25	0.35

This gives rise to a spillover game with $N = \{COM, SOC, CD, LIB\}$ and an N-proper set $\mathcal{W}$ of coalitions having a majority:

$$\begin{aligned}\mathcal{W} &= \{\{SOC, CD\}, \{SOC, LIB\}, \{CD, LIB\}, \{COM, SOC, CD\},\\ &\quad \{COM, SOC, LIB\}, \{COM, CD, LIB\}, \{SOC, CD, LIB\}, N\}\,.\end{aligned}$$

For the winning coalitions the payoffs could look as follows (the first entry in the two-dimensional z^S-vectors corresponds to COM):

S	$v(S)$	z^S
$\{SOC, CD\}$	12	$(4, 3)$
$\{SOC, LIB\}$	10	$(2, 7)$
$\{CD, LIB\}$	15	$(0, 4)$
$\{COM, SOC, CD\}$	19	0
$\{COM, SOC, LIB\}$	13	6
$\{COM, CD, LIB\}$	14	4
$\{SOC, CD, LIB\}$	18	1
N	16	

Obviously, COM and LIB do not have much in common, which is reflected by a relatively low payoff to coalitions in which both are involved. The central position of CD is reflected by the relatively high spillover it experiences when a coalition forms in which it is not involved. If all four parties get together, the resulting coalition will not be very homogeneous, which is reflected by a low value for N.

To construct a marginal vector, assume that first the largest party, LIB, enters. Since this party on itself is not winning, its marginal contribution is zero. To keep things simple, we assume that parties always join if the coalition in place is not yet winning. Hence, the second largest party, SOC, joins, creating a winning coalition.

[1]This example is inspired by the model presented in Van der Rijt (2000).

Its payoff equals the marginal contribution to the existing coalition, which equals 10-0=10. Next, the third largest, CD has the choice whether to join or not. If it joins, its marginal contribution is 18-10=8. If it does not join, the worst that can happen is that coalition $\{COM, SOC, LIB\}$ eventually emerges, giving CD a payoff (spillover) of 6. Hence, CD joins the existing coalition. Finally, COM decides not to join, giving it a spillover of 1 rather than the marginal contribution of -2. So, the resulting coalition will be $\{SOC, CD, LIB\}$ with payoff 1 to COM, 10 to SOC, 8 to CD and 0 to LIB. $\triangleleft$

The procedure described in the previous example resembles the well-known concept of marginal vector for TU games. The crucial difference, however, is that contrary to the TU case, in our context players do not *have to* join the existing coalition. As long as there is a winning coalition in place and the worst that can happen if a player does not join is better than joining, that player has the option to stay outside.

Let $(N, \mathcal{W}, v, z)$ be a spillover game. The *marginal vector corresponding to* σ, $\sigma \in \Pi(N)$, denoted by $M^\sigma(N, \mathcal{W}, v, z)$, is defined recursively. By S_k^σ we denote the current coalition after the first k players have entered and we initialise $S_0^\sigma = \emptyset$. Let $k \in \{1, \dots, n\}$. We assume that player $i = \sigma(k)$ has to join the coalition in place, S_{k-1}^σ, if this coalition is not yet winning. Otherwise, he has to choose between joining and staying out. As a result of monotonicity of $\mathcal{W}$, once a winning coalition is in place, a winning coalition will result regardless of whether the next player joins or not. The minimum payoff to player $i = \sigma(k)$ if he chooses not to join the winning coalition S_{k-1}^σ equals

$$m_i^\sigma = \min_{T \subset N \setminus \{i\} : T \cap P_i^\sigma = S_{k-1}^\sigma} z_i^T,$$

where $P_i^\sigma = \{j \in N \mid \sigma^{-1}(j) < \sigma^{-1}(i)\}$. If he does join, his marginal contribution equals

$$c_i^\sigma = v(S_{k-1}^\sigma \cup \{i\}) - v(S_{k-1}^\sigma).$$

If player i has the choice, he decides not to join S_{k-1}^σ if the worst that can happen to i if he stays outside, m_i^σ, is better than his marginal contribution c_i^σ. So,

$$S_k^\sigma = \begin{cases} S_{k-1}^\sigma & \text{if } S_{k-1}^\sigma \in \mathcal{W} \text{ and } m_i^\sigma > c_i^\sigma, \\ S_{k-1}^\sigma \cup \{i\} & \text{otherwise} \end{cases}$$

and

$$M_i^\sigma(N, \mathcal{W}, v, z) = \begin{cases} z_i^{S_n^\sigma} & \text{if } S_{k-1}^\sigma \in \mathcal{W} \text{ and } m_i^\sigma > c_i^\sigma, \\ c_i^\sigma & \text{otherwise.} \end{cases}$$

According to this procedure, the coalition $S^\sigma = S_n^\sigma$ eventually results and in the corresponding marginal vector, $v(S^\sigma)$ is divided among the members of S^σ and the players in $N \backslash S^\sigma$ get their corresponding spillovers.

The solution that is computed in Example 6.4.1 is the marginal vector that corresponds to the ordering based on the shares of the seats. Of course, this procedure can be performed with all orderings on the parties, each leading to a marginal vector. The *Shapley value* (cf. Shapley (1953)) is defined as the average of these marginal vectors:

$$\Phi(N, \mathcal{W}, v, z) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} M^\sigma(N, \mathcal{W}, v, z).$$

The Shapley value can be interpreted as an *expected* vector of power indices if the orderings on the players are equally likely. The total power according to different marginal vectors need not be the same. Contrary to each marginal vector separately, the Shapley value is not "supported" by a single coalition.

Example 6.4.2 Recall from Example 6.4.1 that the marginal vector corresponding to the order $\sigma = (LIB, SOC, CD, COM)$ equals $M^\sigma = (1, 10, 8, 0)$, with resulting coalition $\{SOC, CD, LIB\}$. If we take the order $\tau = (CD, COM, LIB, SOC)$, we obtain the marginal vector $M^\tau = (0, 4, 0, 14)$ with corresponding coalition $\{COM, CD, LIB\}$. Note that $\sum_{i \in N} M_i^\sigma \neq \sum_{i \in N} M_i^\tau$.

Computing all marginal vectors and taking the average yields the Shapley value:

$$\Phi(N, \mathcal{W}, v, z) = \frac{1}{24}(24, 140, 172, 116).$$

It is readily seen that there exists no coalition for which the Shapley value is an allocation. ◁

The procedure presented in the definition of marginal vector should not be viewed as a description of how governing coalitions are or should be formed. Rather, these marginal vectors are an indication of what could happen and through the Shapley value, they provide an insight into the relative power of the players.

The strategic element in our definition of marginal vector is that a player can choose not to join when it is in his interest to stay separate. We assume that players are cautious in that they only decide not to join when the worst that can happen when doing so is better than the payoff if they join.[2] This strategic element can be extended in several ways. For example, one can assume that the players play a sequential move extensive form game and the resulting marginal vector is the payoff vector corresponding to a subgame perfect equilibrium.

6.5 Public-private connection problems

In many allocation decisions resulting from Operations Research (OR) problems (cf. Borm et al. (2001)), spillovers occur naturally. In this section, we analyse public-private connection (ppc) problems. We address two main questions: which coalition will cooperate and how should the value of this coalition be divided among its members? We conclude this section by indicating how our model can be used to handle spillovers in other OR games.

Consider a group of players that can be connected to a source. If a player is connected to the source, he receives some fixed benefit. On the other hand, by creating connections costs are incurred. Each player can construct a direct link between the source and himself, or he can connect himself via other players.

There are two types of connections: public and private. If a player constructs a public link, other players can use this link to get to the source. A private connection can only be used by the player who constructs it.

When constructing a network, players can cooperate in order to reduce costs. We assume that if a group of players cooperate, the players within that coalition construct an optimal *public* network, which by definition is open for use by other players. Once this optimal public network for the coalition is constructed, the players outside can decide whether or not to connect to the source, using the public network in place, possibly complemented with *private* connections. The corresponding payoffs to these individual players are the spillovers that result from the formation of this coalition. We call the resulting model a public-private connection (ppc) problem. Note that in principle every coalition can build the public network and hence, $\mathcal{W} = 2^N$.

Before formally introducing ppc problems, we start with an example.

[2]This is quite standard practice in cooperative game theory. Usually, a noncooperative game is turned into a TU game by assigning values to coalitions based on the maximin principle, ie, by assuming that the players in a coalition maximise their payoff given that the other players try to minimise this payoff.

Example 6.5.1 Consider the ppc problem depicted in Figure 6.1, where * is the source, the bold numbers indicate the players, the numbers between parentheses represent the benefits if the players are connected to the source and the numbers on the edges are the corresponding construction costs.

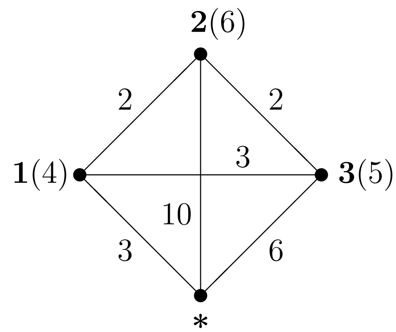


Figure 6.1: A ppc problem

First, consider the grand coalition. The best this coalition can do is to build a public network connecting all players to the source, creating links $\{*, 1\}$, $\{1, 2\}$ and $\{2, 3\}$. The net payoff equals $4 + 6 + 5 - (3 + 2 + 2) = 8$.

Next, consider coalition $\{2\}$.[3] It is optimal for this coalition to create $\{*, 1\}$ and $\{1, 2\}$, giving player 2 a payoff of $6 - (2 + 3) = 1$. The construction of these public links results in spillovers for players 1 and 3. Player 1 can use the public network and does not have to create an extra private link, so his spillover equals his benefit of 4. Player 3 can also use the public network, complemented with the private connection $\{2, 3\}$, giving him a spillover of $5 - 2 = 3$.

Next, consider $\{3\}$. Since every path to the source is more expensive that his benefit, player 3 will not construct a network at all, giving him a payoff of 0. Player 1 then has to construct a private link $\{*, 1\}$ with spillover 1 and player 2, who cannot use 1's private link, will have to construct $\{*, 1\}$ and $\{1, 2\}$ privately, giving him a spillover of 1 as well.

Doing this for every possible coalition, we obtain a spillover game $\mathcal{G} = (N, \mathcal{W}, v, z)$ with $N = \{1, 2, 3\}$, $\mathcal{W} = 2^N$ and the following payoffs:

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v(S)$	0	1	1	0	5	3	4	8
z^S	$(1,1,0)$	$(4,2)$	$(4,3)$	$(1,1)$	3	4	4	

◁

[3]It may seem strange that a single player or even the empty coalition can build a public network. For the sake of expositional clarity, we do not *a priori* exclude this possibility.

A *public-private connection* or *ppc problem* is a triple $(N, *, b, c)$, where $N = \{1, \ldots, n\}$ is a set of agents, $*$ is a source, $b : N \to \mathbb{R}_+$ is a nonnegative benefit function and $c : E_{N^*} \to \mathbb{R}_+$ is a nonnegative cost function, where $N^* = N \cup \{*\}$. E_S is defined as the set of all edges between pairs of elements of $S \subset N^*$, so that (S, E_S) is the complete graph on S:

$$E_S = \{\{i, j\} \,|\, i, j \in S, i \neq j\}.$$

$b(i)$ represents the benefits if player $i \in N$ is connected to $*$ and $c(\{i, j\})$ represents the costs if a link between $i \in N^*$ and $j \in N^*$ is formed.

To avoid unnecessary diversions, we simply assume that the optimal public network for each coalition is unique.

A network of edges is a set $K \subset E_{N^*}$. By $N(K) \subset N$ we denote the set of players that are connected to the source in network K.

A ppc problem $(N, *, b, c)$ gives rise to a *public-private connection game* or *ppc game* $(N, \mathcal{W}, v, z)$ with $\mathcal{W} = 2^N$,

$$v(S) = \max_{K \subset E_{N^*}} \{ \sum_{i \in S \cap N(K)} b(i) - \sum_{k \in K} c(k) \} \tag{6.2}$$

for all $S \subset N$ and

$$z_i^S = \max_{L \subset E_{N^*} \setminus K_S} \{ b(i) I_{N(K_S \cup L)}(i) - \sum_{\ell \in L} c(\ell) \},$$

for all $S \subset N, i \in N \setminus S$, where K_S denotes the unique network K that maximises (6.2), and $I_A(i)$ equals 1 if $i \in A$ and 0 if $i \notin A$.

Although the players outside S can use the public network created by S, the spillovers need not be positive. This is caused by the assumption that only the players within the coalition that eventually builds the public network can cooperate, whereas the players outside can only build private links. As a result, the costs of a particular connection may have to be paid more than once by the players outside the coalition and consequently, they could be worse off than when they cooperate.

Public-private connection games are superadditive, as is shown in the following proposition.

Proposition 6.5.1 *Let $(N, *, b, c)$ be a ppc problem. Then the corresponding game $(N, \mathcal{W}, v, z)$ is superadditive.*

Proof: Let $S \subset T \subset N$. Let K_S be the optimal public network for S and for all $i \in T \backslash S$, let $L_i^{N \backslash S}$ be the optimal private network for i, given that K_S is present. Define $\bar{K} = K_S \cup \bigcup_{i \in T \backslash S} L_i^{N \backslash S}$. Then

$$
\begin{aligned}
v(T) &= \max_{K \subset E_{N^*}} \{ \sum_{i \in T \cap N(K)} b(i) - \sum_{k \in K} c(k) \} \\
&\geq \sum_{i \in T \cap N(\bar{K})} b(i) - \sum_{k \in \bar{K}} c(k) \\
&= \sum_{i \in S \cap N(\bar{K})} b(i) - \sum_{k \in K_S} c(k) + \sum_{i \in (T \backslash S) \cap N(\bar{K})} b(i) - \sum_{k \in \bar{K} \backslash K_S} c(k) \\
&\geq v(S) + \sum_{i \in T \backslash S} b(i) I_{N(\bar{K})}(i) - \sum_{k \in \bar{K} \backslash K_S} c(k) \\
&\geq v(S) + \sum_{i \in T \backslash S} \left[b(i) I_{N(K_S \cup L_i^{N \backslash S})}(i) - \sum_{\ell \in L_i^{N \backslash S}} c(\ell) \right] \\
&= v(S) + \sum_{i \in T \backslash S} z_i^S .
\end{aligned}
$$

□

Although public-private connection games are superadditive, they need not be convex, as is illustrated in the following example.

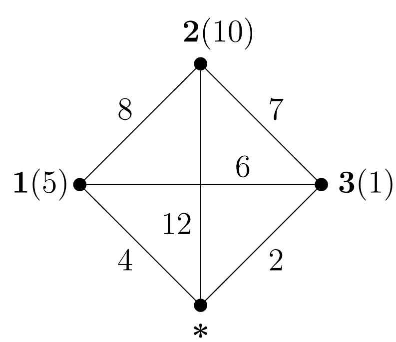


Figure 6.2: A ppc problem

Example 6.5.2 Consider the ppc problem depicted in Figure 6.2. Let $S = \{1\}$, $T = \{1, 3\}$ and $U = \{2\}$. Then for the corresponding game $(N, \mathcal{W}, v, z)$ we have

$$
v(T \cup U) - \sum_{j \in T} z_j^U = v(N) - z_1^{\{2\}} - z_3^{\{2\}}
$$

$$\begin{aligned} &= 3-1-1 \\ &< 3-1 \\ &= v(\{1,2\}) - z_1^{\{2\}} \\ &= v(S \cup U) - \sum_{j \in S} z_j^U. \end{aligned}$$

Hence, this game is not (coalition merge) convex. ◁

Let us return to the ppc problem in Example 6.5.1. To find a suitable solution for this problem, we first consider the core of the corresponding ppc game $\mathcal{G}$. The S-cores are given in the following table.

S	$C_S(\mathcal{G})$
$\emptyset$	$\emptyset$
$\{1\}$	$\{(1,4,2)\}$
$\{2\}$	$\{(4,1,3)\}$
$\{3\}$	$\emptyset$
$\{1,2\}$	$Conv(\{(4,1,3),(1,4,3)\})$
$\{1,3\}$	$Conv(\{(3,4,0),(1,4,2)\})$
$\{2,3\}$	$Conv(\{(4,4,0),(4,1,3)\})$
N	$Conv(\{(4,1,3),(4,4,0),(3,5,0),(1,5,2),(1,4,3)\})$

Since the N-core (weakly) dominates all the other cores, we have $C(\mathcal{G}) = C_N(\mathcal{G})$. Note that there are some core elements that are supported by other coalitions as well, all of which contain player 2. The core element $(4,1,3)$ is even supported by every coalition containing player 2.

In Figure 6.3 we depict the four S-cores that yield core elements and (therefore) lie in the hyperplane with total payoff 8. The payoff to player 1 is in normal typeface, the payoff to player 2 is italic and the payoff to player 3 is bold. The N-core $C_N(\mathcal{G})$ is the shaded pentagon, $C_{\{1,2\}}$ is the line segment with the triangles, $C_{\{2,3\}}$ is the line segment with the stars and $C_{\{2\}}$ is the point $(4,1,3)$.

To solve the ppc problem, suppose for the moment that all players cooperate. We have already seen that it is optimal for the grand coalition to connect all its members to the source. Since the benefits of a coalition do not depend on the shape of the network that is formed as long as everyone is connected, the optimal network in this ppc problem, $\{\{*,1\},\{1,2\},\{2,3\}\}$, is actually a *minimum cost spanning tree* (cf. Claus and Kleitman (1973)). In this context, Bird (1976) proposed that each player pays the costs of the (unique) link that is adjacent to him and lies on the path between him and the source. So, one way to solve a ppc problem is to

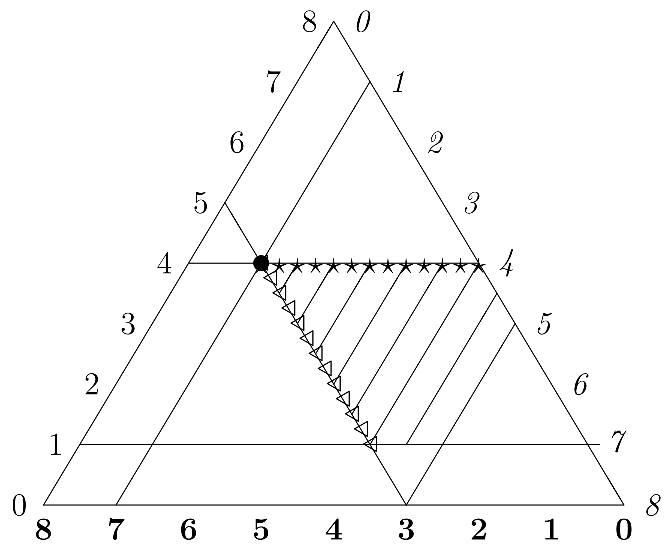


Figure 6.3: The core of $\mathcal{G}$

assume that construction costs are divided using Bird's rule and everyone gets his own benefit. According to this Bird-like procedure, player 1 receives $4 - 3 = 1$, player 2 gets $6 - 2 = 4$ and player 3 gets $5 - 2 = 3$. This yields the core element $(1, 4, 3)$ as solution.

This procedure, however, has some elementary flaws. The nice properties of the Bird rule for minimum cost spanning tree problems follow from the assumption that all players *have to* cooperate and connect to the source. Moreover, this rule does not take the spillovers into account. The strategic option of players not to participate in a coalition undermines the Bird approach. Player 1 will never agree to the proposed payoff vector $(1, 4, 3)$, since he will be better off leaving the grand coalition, which will lead to a payoff (spillover) of 4. Knowing this, player 3 can argue that he should at least receive 3, his spillover when player 2 forms a coalition on his own. Taking this into account, the payoff vector $(4, 1, 3)$ seems a more reasonable outcome. Because player 2 on his own will build a network that also connects player 1 to the source, the latter player occupies a position of power in this ppc problem, which should somehow be reflected in his payoff.

The payoff vector $(4, 1, 3)$ is a core element of the corresponding ppc game and is supported by all coalitions containing player 2. This payoff, however, is not acceptable to player 2. He can argue that if he were to refuse to build his optimal public network, it would then be optimal for players 1 and 3 to work together, giving player 2 a spillover of 4.

By considering this kind of strategic threats of the players not to cooperate, any seemingly reasonable proposal can be dismissed. As a result, it is not clear which coalition will eventually emerge and what the corresponding payoffs will be.

This phenomenon of *free-riding* is well-known in the context of public goods. Although it is socially optimal for all the players to cooperate in order to provide a public good, the players separately have the strategic incentive not to do so.

One way to solve this problem is to apply the Shapley value, as defined in the previous section. In each marginal vector, the strategic aspects mentioned above are taken into account. By averaging over all marginal vectors, some kind of "average" influence of these noncooperative considerations is reflected in the payoff.

Example 6.5.3 Consider the ppc problem of Example 6.5.1. The Shapley value equals

$$\Phi(N, \mathcal{W}, v, z) = \frac{1}{6}(17, 20, 11).$$

$\triangleleft$

As was discussed in the previous section, the definition of marginal vector can be adapted to reflect the level of strategic considerations one wants to incorporate in the model.

Public-private connection situations are not the only class of OR problems in which spillovers occur. A related phenomenon arises in *travelling salesman situations* (cf. Tamir (1989)). In a travelling salesman situation, there is a graph in which the vertices represent the locations of the players (and the salesman) and the edges represent the roads between them along which the salesman can travel. The problem is to find a cheapest Hamiltonian circuit in this graph, where each edge has a nonnegative cost associated with it.

Also, each subcoalition faces the same problem of finding a cheapest Hamiltonian circuit through the vertices in which the players in this coalition and the salesman are located. This gives rise to a cooperative cost game. As is the case in minimum cost spanning tree situations, however, one does not take into account that there are spillovers involved. If a subcoalition of players decides to work together and invite the salesman to travel to them according to their cheapest tour, the salesman might come near some players outside the coalition, making it cheaper for them to have him come to visit them as well.

In *sequencing situations* (see section 4.3), spillovers can also play a role. In a sequencing situation, there is a queue of players waiting to be served. The players in the queue might have different opportunity costs, so moving high-cost players to the front while compensating the low-cost players through side payments can result in a Pareto improvement.

Normally, in such situations, only pairs of players who are adjacent in the queue are allowed to switch, so that a third player can never suffer. If we use our spillover model, however, this restriction is unnecessary, since the effect of any pairwise switch on the other players can be taken into account explicitly.

Chapter 7

Bankruptcy situations with references

7.1 Introduction

Bankruptcy problems were introduced by O'Neill (1982) and have been subsequently analysed in a variety of contexts. In a *bankruptcy situation*, one has to divide a given amount of money (*estate*) amongst a set of agents, each of whom has a *claim* on the estate. The total amount claimed typically exceeds the estate available, so not all the claims of the agents can be fully satisfied.

The example originally given by O'Neill (and which is inspired by some passages in the Talmud) is that of a bequest: a man dies, leaving behind an estate which is not sufficiently large to satisfy all promises made to his heirs in his *will*. Another example is that of a firm going bankrupt, whose assets are insufficient to satisfy all creditors' outstanding claims.

O'Neill proposes a particular solution to this problem, which he calls the method of recursive completion (also known as the run-to-the-bank rule). This solution turns out to be the Shapley value of a corresponding *bankruptcy game*, which is a transferable utility game where the value of each coalition is the amount of money that is left of the estate after all the claims of the agents outside that coalition are satisfied. Aumann and Maschler (1985) and Curiel et al. (1987) proposed and characterised two further solutions that coincide with the nucleolus and compromise value of the corresponding bankruptcy game, respectively.

O'Neill's bankruptcy model has been applied to a wide array of economic problems, eg, taxation problems (cf. Young (1988)), surplus-sharing problems (cf. Moulin (1987)), cost-sharing problems (cf. Moulin (1988)), apportionment of indivisible good(s) problems (cf. Young (1994)) and priority problems (cf. Moulin (2000) and Young (1994)).

In some situations the claims of the players are not the only quantities that are relevant for determining how to divide the estate. Pulido et al. (2002) analyse the problem of dividing a sum of money to the various degree courses that are offered at Miguel Hernandez University in Elche, Spain. Each course has a claim, which reflects in some way the monetary *needs* of this course. These needs are determined within a fixed set of rules and are verifiable to everyone involved. In addition to these claims, the Valencian government (*Generalitat Valenciana*) provides a set of rules of its own to indicate what each course should get, without taking into account how much money is available. This allocation can be considered as an exogenous *reference point* for determining a fair division of the estate.

Clearly, both the claims and the references form a relevant basis for the allocation decision. The natural question is how these two criteria in such a *bankruptcy situation with references* should be combined in order to reach a fair outcome. In Pulido et al. (2002), the special case is considered, in which the estate suffices to implement the reference point. In this case, the references can be interpreted as *rights*. They describe a two-stage procedure which first gives each claimant his reference amount and then shares the remainder using a bankruptcy rule.

In this chapter, which is based on Pulido et al. (2003), we consider situations in which the estate is not necessarily big enough to pay all reference amounts. Hence, we do not regard these references as rights. As a result, our analysis extends the analysis in Pulido et al. (2002), but we provide different answers on the class of situations in which both models are applicable.

We consider two ways in which the claim and reference vectors are combined and for either approach, we define a compromise solution. The underlying idea is the following: for each player we combine the claim and reference vectors in such a way that the resulting payoff to him is maximal. Doing this for every player, we obtain an upper vector, which can be seen as a utopia point. On the other hand, we find for each player that combination which gives him a minimal outcome, which results in a lower vector. The compromise solution is then defined as the convex combination of the upper and lower vector that is efficient with respect to the estate.

For both approaches we also define a corresponding *bankruptcy game with references*. These games are exact, but not necessarily convex. Our two compromise solutions turn out to coincide with the compromise values of these games.

This chapter is organised as follows. First, in section 2 we introduce bankruptcy situations and discuss some bankruptcy rules. In section 3, we introduce bankruptcy situations with references. In section 4 we present our compromise solutions. In section 5 we define and analyse the two corresponding games and show that the compromise solutions coincide with their compromise values.

7.2 Bankruptcy situations and games

A *bankruptcy situation* (cf. O'Neill (1982)) is a triple (N, E, c), where $N = \{1, \ldots, n\}$ is the set of players, $E \geq 0$ is the estate to be divided and $c \in \mathbb{R}^N_{++}$ is the vector of claims such that $C \geq E$, where we define $C = c(N)$.[1] We denote the class of bankruptcy situations with player set N by BR^N. As with games, we sometimes omit the player set and denote a bankruptcy situation by (E, c).

A *bankruptcy rule* is a function $f : BR^N \to \mathbb{R}^N$ that assigns to every bankruptcy situation $(N, E, c) \in BR^N$ a payoff vector $f(N, E, c) \in \mathbb{R}^N$ such that

$$\begin{array}{ll} 0 \leq f_i(N,E,c) \leq c_i \text{ for all } i \in N & \text{(reasonability)}, \\ \sum_{i \in N} f_i(N,E,c) = E & \text{(efficiency)}. \end{array}$$

In the literature, many bankruptcy rules have been proposed. The most well-known are summarised below.

- *Proportional rule*: $PROP(N, E, c) = \frac{E}{C}c$, ie, each player gets a share of E proportional to his claim.

- *Constrained equal award rule*: $CEA_i(N, E, c) = \min\{\alpha, c_i\}$ for all $i \in N$, where $\alpha \in \mathbb{R}$ is such that $\sum_{i \in N} CEA_i(N, E, c) = E$, ie, each player receives the same amount, provided that this does not exceed his claim.

- *Constrained equal loss rule*: $CEL_i(N, E, c) = \max\{c_i - \beta, 0\}$, where $\beta \in \mathbb{R}$ is such that $\sum_{i \in N} CEL_i(N, E, c) = E$, ie, each player loses the same amount with respect to his claim, provided that he gets at least zero.

- *Talmud rule* or *contested garment rule* (Aumann and Maschler (1985)):

$$TAL(N, E, c) = \begin{cases} CEA(N, E, \frac{1}{2}c) & \text{if } C \geq 2E, \\ c - CEA(N, C - E, \frac{1}{2}c) & \text{if } C < 2E. \end{cases}$$

[1] Throughout this chapter, for a vector $x \in \mathbb{R}^N$ we denote $x(S) = \sum_{i \in S} x_i$ for all $S \subset N$.

If the estate is small, then this amount is divided using the CEA rule after halving the claims. Otherwise each player receives his claim and the difference is taken back using CEA.

- *Run-to-the-bank-rule* or *random arrival rule* or *recursive completion rule* (cf. O'Neill (1982)): $RTB(N,E,c) = \frac{1}{n!}\sum_{\sigma\in\Pi(N)} \rho(\sigma)$, where for all $\sigma \in \Pi(N)$ and $p \in \{1,\ldots n\}$, $\rho_{\sigma(p)}(\sigma) = \max\{\min\{c_{\sigma(p)}, E - \sum_{k=1}^{p-1} c_{\sigma(k)}\}, 0\}$. So an ordering σ determines a race to the bank, where the players arriving at the bank receive their claim as long as there is still some money available. The run-to-the-bank solution is the average over all such races.

- *Adjusted proportional rule* (cf. Curiel et al. (1987)): $APROP(E,c) = m(E,c) + PROP(E',c')$, where $m_i(E,c) = \max\{E - \sum_{j\in N\setminus\{i\}} c_j, 0\}$ denotes player i's minimal right, $E' = E - \sum_{i\in N} m_i(E,c)$ and for all $i \in N$, $c_i' = \min\{c_i - m_i(E,c), E'\}$. First, each player receives his minimal right and the remainder is divided using the proportional rule, where each player's claim is truncated to the estate left.

Every bankruptcy situation $(N,E,c) \in BR^N$ gives rise to a *bankruptcy game* $v_{E,c} \in TU^N$, where the value of a coalition $S \subset N$ is given by

$$v_{E,c}(S) = \max\{E - \sum_{i\in N\setminus S} c_i, 0\}. \tag{7.1}$$

So $v_{E,c}(S)$ is that part of the estate that is left for the players in S after the claims of all the other players have been satisfied.

A bankruptcy rule f is called *game-theoretic* if for all $(N,E,c), (N,E,c') \in BR^N$ such that $v_{E,c} = v_{E,c'}$ we have $f(N,E,c) = f(N,E,c')$. Curiel (1988) shows that f is game theoretic if and only if it satisfies the truncation property, ie, for all $(N,E,c) \in BR^N$, $f(N,E,c) = f(N,E,c')$ with $c_i' = \min\{c_i, E\}$ for all $i \in N$. Of the rules discussed in this section, only $PROP$ and CEL are not game-theoretic.

Some bankruptcy solutions turn out to coincide with well-known solutions of the corresponding bankruptcy games. O'Neill (1982) showed that $RTB(N,E,c) = \Phi(v_{E,c})$ for all $(N,E,c) \in BR^N$. Similarly, the Talmud rule coincides with the nucleolus of the corresponding game (cf. Aumann and Maschler (1985)) and the adjusted proportional rule coincides with the compromise value (cf. Curiel et al. (1987)).

A nice survey on bankruptcy situations is provided by Thomson (2003).

7.3 Bankruptcy situations with references

A *bankruptcy situation with references* (cf. Pulido et al. (2002)) is a 4-tuple (N, E, r, c), where $N = \{1, \ldots, n\}$ is the set of players, $E \geq 0$ is the estate under contest, $r \in \mathbb{R}^N_+$ is the vector of references and the vector of claims, $c \in \mathbb{R}^N_{++}$, is such that $c(N) \geq E$. The claim vector c has the same interpretation as the claim vector in standard bankruptcy situations, while r represents some exogenously given reference point for the division of the estate. We assume that $r_i \leq c_i$ for every player $i \in N$. We denote $R = r(N)$ and $C = c(N)$. The set of all bankruptcy situations with references with player set N is denoted by BRR^N.

Pulido et al. (2002) distinguish between two types of bankruptcy situations with references: *CERO bankruptcy situations* ($C \geq E \geq R \geq 0$) and *CREO bankruptcy situations* ($C \geq R > E \geq 0$). They only analyse the CERO case, in which the estate is sufficient to give each player his reference amount. As pointed out in the introduction, in the CERO case the references can be interpreted as *rights*. Basically, such a situation can be solved by first allocating this reference point and then dividing the surplus $E - R$. Using this idea, Pulido et al. (2002) define corresponding *CERO bankruptcy games*.

We consider both cases simultaneously, so in our context the references cannot necessarily all be satisfied and can therefore not be considered as rights. Hence, we take a different approach to solving such situations and defining appropriate corresponding games. As a result, the analysis differs from Pulido et al. (2002) even on the class of CERO situations.

In order to come to a solution we will have to make some assumptions on the way in which the claims and reference point are used to divide the money. Obviously, the claim and reference vectors should both be taken into account, but this can be done in a number of ways. We first construct a new "demand" vector[2], reflecting both r and c. After that, a given bankruptcy rule f is applied to this new vector.

Throughout our analysis, we consider two approaches to constructing the combined demand vector. For either approach we define a *family* of compromise solutions and games, depending on the choice for f.

We assume that f satisfies *complementary monotonicity (CM)*: for all $S \subset N$

[2]To avoid confusion, in the remainder of this chapter we will reserve the term *claim vector* for the bankruptcy situation with references. The claims in the ensuing bankruptcy situation are called *demands*, which we denote by the demand vector $d \in \mathbb{R}^N_+$.

and $d, d' \in \mathbb{R}^N_+$ such that $d_j = d'_j$, for all $j \in N \backslash S$ and $d_i \leq d'_i$ for all $i \in S$, we have that $f_j(N, E, d) \geq f_j(N, E, d')$ for all $j \in N \backslash S$. Using an induction argument, it is easily established that this is equivalent to the same requirement only for all one-person coalitions S.

All bankruptcy rules mentioned in section 7.2 satisfy CM, with the notable exception of $APROP$, as is shown in Pulido (2001).

7.4 Compromise solutions

In this section, we define two (families of) compromise solutions for bankruptcy situations with references. As stated in the previous section, we assume that first the reference and claim vectors are combined into a new demand vector. But instead of directly combining these two vectors, we determine, given the rule f, for each player which combination leads to the highest payoff to him and which one to his lowest payoff. This leads to an upper and lower bound for the allocation of the estate. The compromise solution is then simply defined as the unique efficient convex combination of these two vectors.

Geometrically, combining claims and references boils down to picking a point in the hypercube $\Pi_{i \in N}[r_i, c_i]$. We consider two possibilities. In our first approach, the *extreme approach*, we consider the extreme points of this hypercube, ie, points in which some players demand their reference amount and the others their claim. The lower vector ℓ^f and the upper vector L^f are defined by

$$\ell^f_i(N, E, r, c) = \begin{cases} f_i(N, E, (r_i, c_{N \backslash \{i\}})) & \text{if } r_i + c(N \backslash \{i\}) \geq E, \\ E - c(N \backslash \{i\}) & \text{if } r_i + c(N \backslash \{i\}) < E \end{cases}$$

and

$$L^f_i(N, E, r, c) = \begin{cases} f_i(N, E, (c_i, r_{N \backslash \{i\}})) & \text{if } c_i + r(N \backslash \{i\}) \geq E, \\ c_i & \text{if } c_i + r(N \backslash \{i\}) < E \end{cases}$$

for all $i \in N$. It follows from CM that $(r_i, c_{N \backslash \{i\}})$ is the worst extreme point for player i and that $(c_i, r_{N \backslash \{i\}})$ is the best. If in a point the estate suffices to satisfy all demands, then player i gets what is left by the other players, with a maximum of his own claim c_i.

In the following lemma, which we will prove in section 5, we show that ℓ^f and L^f can indeed be considered as lower and upper bounds, respectively, for the division of the estate.

Lemma 7.4.1 *Let $(N, E, r, c) \in BRR^N$ be a bankruptcy situation with references. Then for the corresponding lower and upper vectors we have $\ell^f \leq L^f$ and $\sum_{i \in N} \ell_i^f \leq E \leq \sum_{i \in N} L_i^f$.*

The *extreme compromise solution* $\gamma^f : BRR^N \to \mathbb{R}^N$ is defined by

$$\gamma^f = \alpha \ell^f + (1-\alpha) L^f,$$

where $\alpha \in [0,1]$ is such that $\sum_{i \in N}(\alpha \ell_i^f + (1-\alpha) L_i^f) = E$. As a result of the previous lemma, such α exists.

In our second approach, the *diagonal approach*, we consider the main diagonal of the hypercube. The lower and upper vectors $\bar{\ell}^f$ and $\bar{L}^f$ are defined by

$$\begin{aligned} \bar{\ell}_i^f(N, E, r, c) &= \inf_{\lambda \in [0,1]} h_i^{f,\lambda}(N, E, r, c), \\ \bar{L}_i^f(N, E, r, c) &= \sup_{\lambda \in [0,1]} h_i^{f,\lambda}(N, E, r, c), \end{aligned}$$

where for all $i \in N$ and $\lambda \in [0,1]$,

$$h_i^{f,\lambda}(N, E, r, c) = \begin{cases} f_i(N, E, \lambda r + (1-\lambda)c) & \text{if } \lambda R + (1-\lambda)C \geq E, \\ \lambda r_i + (1-\lambda)c_i + f_i(N, \overline{E}^\lambda, \overline{d}^\lambda) & \text{if } \lambda R + (1-\lambda)C < E \end{cases}$$

with $\overline{E}^\lambda = E - (\lambda R + (1-\lambda)C)$ and $\overline{d}^\lambda = c - (\lambda r + (1-\lambda)c) = \lambda(c - r)$. Also in the diagonal case, the vectors $\bar{\ell}$ and $\bar{L}$ can be considered as lower and upper bounds, as is shown for the extreme approach in Lemma 7.4.1.

The *diagonal compromise solution* $\bar{\gamma}^f$ is defined by

$$\bar{\gamma}^f = \alpha \bar{\ell}^f + (1-\alpha) \bar{L}^f,$$

where $\alpha \in [0,1]$ is such that $\sum_{i \in N}(\alpha \bar{\ell}_i^f + (1-\alpha) \bar{L}_i^f) = E$.

7.5 Bankruptcy games with references

In this section, we define for either approach a corresponding cooperative game. In line with Pulido et al. (2002), we take a pessimistic point of view, so the definitions of the characteristic functions resemble the ones for the lower vectors in the previous section. Throughout the analysis, we assume that f is a CM bankruptcy rule.

We define the *extreme game* $v^f \in TU^N$ by

$$v^f(S) = \begin{cases} \sum_{i\in S} f_i(N, E, (r_S, c_{N\backslash S})) & \text{if } r(S) + c(N\backslash S) \geq E, \\ E - c(N\backslash S) & \text{if } r(S) + c(N\backslash S) < E \end{cases}$$

for all $S \subset N$. Note that if $r(S) + c(N\backslash S) < E$, the ensuing problem is not a bankruptcy situation and the players in S can obtain what is left by the players in $N\backslash S$.

As a result of CM, it immediately follows that $(r_S, c_{N\backslash S})$ is the worst point for S in the hypercube, so $v^f(S)$ actually represents the most pessimistic situation for coalition S under the extreme approach. Although it is intuitively clear that this should be the worst point for S, we need CM to ensure that it is actually so.[3]

The *diagonal game* is defined by $\bar{v}^f(S) = \inf_{\lambda\in[0,1]} \bar{v}^f_\lambda(S)$, where

$$\bar{v}^f_\lambda(S) = \begin{cases} \sum_{i\in S} f_i(N, E, \lambda r + (1-\lambda)c) & \text{if } \lambda R + (1-\lambda)C \geq E, \\ \lambda r(S) + (1-\lambda)c(S) + \sum_{i\in S} f_i(N, \overline{E}^\lambda, \overline{d}^\lambda) & \text{if } \lambda R + (1-\lambda)C < E \end{cases}$$

with $\overline{E}^\lambda = E - (\lambda R + (1-\lambda)C)$ and $\overline{d}^\lambda = c - (\lambda r + (1-\lambda)c) = \lambda(c-r)$.

In the second part of the definition the agents receive what is prescribed by the weighted vector and the remainder $\bar{E}^\lambda$ is distributed according to the rule f, using the residual demands $\bar{d}^\lambda$. Note that if (N, E, r, c) is a CREO bankruptcy situation, then in the definitions of both v^f and $\bar{v}^f$ only the first case arises.

As a result of CM, it is readily seen that the game v^f is more pessimistic than $\bar{v}^f$, ie, $v^f(S) \leq \bar{v}^f(S)$ for all $S \subset N$.

The extreme and diagonal games corresponding to a bankruptcy situation with references turn out to be exact. A game $v \in TU^N$ is called *exact* if for all $S \subset N$ there exists an $x \in C(v)$ such that $\sum_{i\in S} x_i = v(S)$. Driessen and Tijs (1985) show that exactness is weaker than convexity and stronger than superadditivity.

Proposition 7.5.1 *Let $(N, E, r, c) \in BRR^N$ be a bankruptcy situation with references. Then the two corresponding games v^f and $\bar{v}^f$ are exact.*

Proof: First, we prove exactness of v^f. Let $S \subset N$ and distinguish between two cases:

1. If $r(S) + c(N\backslash S) \geq E$, then $v^f(S) = \sum_{i\in S} f_i(N, E, (r_S, c_{N\backslash S}))$. Consider $x = f(N, E, (r_S, c_{N\backslash S}))$, so $x(S) = v^f(S)$. It is easily checked that, because f

[3] Pulido (2001) shows that for the adjusted proportional rule, which is not CM, this is not the case.

satisfies CM, $x \in C(v^f)$.

2. If $r(S) + c(N \backslash S) < E$, then $v^f(S) = E - c(N \backslash S)$. Let $x \in \mathbb{R}^N$ be defined by

$$x_i = \begin{cases} c_i & \text{if } i \in N \backslash S, \\ r_i + f_i(S, E - c(N \backslash S) - r(S), c_S - r_S) & \text{if } i \in S \end{cases}$$

for all $i \in N$. Obviously, $x(N) = E = v^f(N)$ and $x(S) = E - c(N \backslash S) = v^f(S)$. Let $T \subset N$ and distinguish between two cases:

(a) If $r(T) + c(N \backslash T) \geq E$, then $v^f(T) = \sum_{i \in T} f_i(N, E, (r_T, c_{N \backslash T})) \leq r(T) \leq x(T)$.

(b) If $r(T) + c(N \backslash T) < E$, then $v^f(T) = E - c(N \backslash T)$. Define $T_1 = T \cap (N \backslash S)$ and $T_2 = T \cap S$. Then,

$$\begin{aligned} x(T) &= c(T_1) + r(T_2) + \sum_{i \in T_2} f_i(S, E - c(N \backslash S) - r(S), c_S - r_S) \\ &\geq c(T_1) + r(T_2) + E - c(N \backslash S) - r(S) - \sum_{i \in S \backslash T_2} (c_i - r_i) \\ &= E + c(T_1) - c(N \backslash S) - c(S \backslash T_2) + r(T_2) + r(S \backslash T_2) - r(S) \\ &= E + c(T_1) - c(N \backslash T_2) = E - c(N \backslash T) = v^f(T). \end{aligned}$$

Hence, $x \in C(v^f)$.

To show that $\bar{v}^f$ is exact, let $S \subset N$ and let $\lambda^* \in [0, 1]$ be such that $\bar{v}^f(S) = \inf_{\lambda \in [0,1]} \bar{v}^f_\lambda(S) = \bar{v}^f_{\lambda^*}(S)$. We distinguish between two cases:

1. If $\lambda^* R + (1 - \lambda^*)C \geq E$, then $x = f(N, E, \lambda^* r + (1 - \lambda^*)c) \in C(\bar{v}^f)$ and $x(S) = \bar{v}^f(S)$.

2. If $\lambda^* R + (1 - \lambda^*)C < E$, then $x = \lambda^* r + (1 - \lambda^*)c + f(N, E - \lambda^* R - (1 - \lambda^*)C, \lambda^*(c - r)) \in C(\bar{v}^f)$ and $x(S) = \bar{v}^f(S)$.

Note that complementary monotonicity of f is not necessary to establish exactness of $\bar{v}^f$. $\square$

The next example shows that the games v^f and $\bar{v}^f$ need not be convex (see section 3.2).

Example 7.5.1 Consider the bankruptcy situation with references $(N, E, r, c) \in BRR^N$ with $N = \{1, 2, 3, 4\}$, $E = 12$, $r = (1, 2, 4, 7)$ and $c = (3, 3, 5, 10)$. Taking the CEA rule, we obtain

S	$\{3\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v^{CEA}(S)$	3	5	$5\frac{1}{2}$	7
$\overline{v}^{CEA}(S)$	3	5	6	7

Then,

$$\begin{aligned} v^{CEA}(\{1,3\}) + v^{CEA}(\{2,3\}) &> v^{CEA}(\{3\}) + v^{CEA}(\{1,2,3\}), \\ \overline{v}^{CEA}(\{1,3\}) + \overline{v}^{CEA}(\{2,3\}) &> \overline{v}^{CEA}(\{3\}) + \overline{v}^{CEA}(\{1,2,3\}). \end{aligned}$$

Therefore, (3.1) is violated and v^{CEA} and $\overline{v}^{CEA}$ are not convex. ◁

Using exactness of v^f, we can now prove Lemma 7.4.1.

Proof of Lemma 7.4.1 From complementary monotonicity of f, $\ell^f \leq L^f$ readily follows. For the second statement, first observe that $\ell_i^f = v^f(\{i\})$ for all $i \in N$. Then exactness of v^f implies

$$\sum_{i \in N} \ell_i^f = \sum_{i \in N} v^f(\{i\}) \leq v^f(N) = E.$$

If $i \in N$ is such that $c_i + r(N \backslash \{i\}) < E$, then $L_i^f = c_i \geq f_i(N, E, c)$. On the other hand, if $c_i + r(N \backslash \{i\}) \geq E$, then $L_i^f = f_i(N, E, (c_i, r_{N \backslash \{i\}})) \geq f_i(N, E, c)$. Combining these two cases, we have $\sum_{i \in N} L_i^f \geq \sum_{i \in N} f_i(N, E, c) = E$. □

The extreme compromise solution coincides with the compromise value of the extreme game v^f, as is shown in the following theorem.

Theorem 7.5.2 *Let $(N, E, r, c) \in BRR^N$ be a bankruptcy situation with references and let f be a complementary monotonic bankruptcy rule. Then $\gamma^f = \tau(v^f)$.*

Proof: First note that as a result of exactness, v^f is compromise admissible. Driessen and Tijs (1985) show that for each exact game $v \in TU^N$, $m_i(v) = v(\{i\})$ for all $i \in N$. Hence, $m_i(v^f) = v^f(\{i\}) = \ell_i^f$. For the upper vector, we have

$$\begin{aligned} L_i^f &= f_i(N, E, (c_i, r_{N \backslash \{i\}})) \\ &= E - \sum_{j \in N \backslash \{i\}} f_j(N, E, (c_i, r_{N \backslash \{i\}})) \\ &= v^f(N) - v^f(N \backslash \{i\}) \end{aligned}$$

if $c_i + r(N\backslash\{i\}) \geq E$ and

$$\begin{aligned} L_i^f &= c_i \\ &= E - (E - c_i) \\ &= v^f(N) - v^f(N\backslash\{i\}) \end{aligned}$$

otherwise. Hence, $L^f = M(v^f)$. From this, we conclude that $\gamma^f = \tau(v^f)$. □

Similarly, one can prove the analogous result for the diagonal compromise solution.

Theorem 7.5.3 *Let $(N, E, r, c) \in BRR^N$ be a bankruptcy situation with references and let f be a complementary monotonic bankruptcy rule. Then $\bar{\gamma}^f = \tau(\bar{v}^f)$.*

Chapter 8

Multi-issue allocation situations

8.1 Introduction

In Chapter 7 we presented the model of bankruptcy situations and extended this model by adding an extra element, a vector of references. In this chapter, we take a different view on bankruptcy problems and introduce the concept of issues.

Generally speaking, a bankruptcy model relates to a particular kind of allocation problem. An *allocation problem* arises whenever a bundle of goods (resources, rights, costs, burdens) is held in common by a group of individuals and must be allotted to them individually. An allocation situation has two ingredients: the goods to be distributed and the claimants amongst whom they are to be allotted. Young (1994) introduced a general framework with the central concept of a "type" of a claimant: "The type of a claimant is a complete description of the claimant for purposes of the allocation, and determines the extent of a claimant's entitlement to the good". A type of a claimant therefore involves a complete description of the claimant in several dimensions or attributes. These attributes are accepted as the benchmark against which allocations are to be judged and can take on many forms, depending on the particular allocation situation at hand. Eg, the allocation of public housing typically depends on such attributes as financial need, family size and time spent on a waiting list. Looking from this general point of view, one can say that the bankruptcy model deals with all allocation problems in which there is one perfectly divisible good (money) to be allocated and the type of each agent can be characterised by a single (monetary) claim on that good.

In a general rationing framework, Kaminski (2000) considers bankruptcy situations in which the type of each claimant is not one-dimensional, as is the case in

O'Neill, but multi-dimensional. In the environment he presents, a type is a vector of claims, the components of which have different legal statuses. As a result, different priorities are assigned to the various components of an agent's claim vector.

The model we present in this chapter, which is based on Calleja et al. (2001), also characterises the types of the claimants in a multi-dimensional way by means of a vector of claims. Contrary to Kaminski however, the multidimensionality of claims is not the necessary consequence of some exogenously given priorities. Our model is inspired by O'Neill's representation of a standard bankruptcy problem in terms of wills. In his context, a *will* is a document stating how much of the estate should go to one particular person. Hence, there is a one-to-one correspondence between the claims of the agents and the wills. Furthermore, it is assumed that all wills are equally "valid" and have the same legal status. The motivation for having as many wills as there are claims is provided by the Talmudic scholar Ibn Ezra. O'Neill hints at generalising Ibn Ezra's approach, stating that "there is no reason why the problem should be restricted in this way". Indeed, why can't a single person be mentioned in more than one will or why should a will only mention one single claimant? We model this kind of situation by considering multi-dimensional claims, the components of which correspond to the wills.

In Young's terminology, the type of a claimant is represented by his claim vector and each will is an attribute. Leaving behind O'Neill's story, we regard each claim component as originating from a particular issue (of which a will is a special case). An *issue* constitutes a *reason on the basis of which the estate is to be divided.* Crucially, such a reason should be well founded and be accepted as such by all parties involved and there should be no a priori discrimination between the issues.

To illustrate the terminology of our model, consider the following example. The central government has to decide how to allocate the taxpayers' money to various public services. The system of government is such that it doesn't allocate this money directly to these services, but indirectly through various government departments. Each department (agent/player) has a number of claims on the amount of money available (estate), arising from those public services (issues) for which it has responsibility. Some of these services are provided by just a single department (eg, tax collection by the Department of Finance), while more departments may be responsible for other services (eg, foreign trade by the Departments of Economic Affairs, Foreign Affairs and Defence). If we were to add up all the claims of a department into one single claim, an ordinary bankruptcy problem would arise. In this bankruptcy problem a rule can be applied to generate an allocation. As argued before, the underlying issues should play a role in determining an outcome. If the departmental claims are combined, however, this crucial information is lost and hence, in our model we take the distinction between the issues explicitly into account.

Another multi-dimensional extension of the bankruptcy model is provided by Lerner (1998). In that paper, a pie is allocated amongst groups, not necessarily disjoint, rather than users.

An interesting application of our model of multi-issue allocation situations can be found in Wintein (2002) and Wintein et al. (2002), where so-called multiple fund investment situations are considered. Given certain restrictions, players have to decide in which funds they invest their capital. This results in a bankruptcy-like model, which is solved using a linear production approach (cf. Owen (1975)). Moreover, an alternative way of looking at this type of investment problem is offered by considering the funds as issues and translating the players' investment opportunities into claims.

The outline of this chapter is as follows. In section 2, we introduce multi-issue allocation situations and define two corresponding cooperative games. These games are constructed from a pessimistic point of view, as are standard bankruptcy games. In order to determine the value of a particular coalition, we let the players outside that coalition decide in which order the issues are to be addressed.

One important assumption in our framework is that once we start paying out money according to one particular issue, this issue must first be fully dealt with before we move on to the next. Going back to O'Neill's example, it seems natural to execute wills completely in a consecutive way rather than satisfying parts of different wills. But this still leaves some freedom within each issue: in our first game (called $\underline{\text{P}}$roportional game), we distribute the money within each issue proportional to the claims in that issue, while in the second game (called $\underline{\text{Q}}$ueue game), we take an even more pessimistic view by allowing the players outside the coalition to choose also the order in which the claims within each issue are satisfied.

The computation of the second of our multi-issue allocation games turns out to be a less than straightforward combinatorial optimisation problem. In section 3, we provide algorithms to determine the worth of coalitions in both approaches.

Properties of multi-issue allocation games are presented in section 4. The main result is that the class of multi-issue allocation games coincides with the class of nonnegative exact games.

In section 5, we analyse run-to-the-bank rules as solutions for multi-issue allocation situations. These rules are based on the interpretation behind the method of recursive completion for bankruptcy situations (cf. O'Neill (1982)). As the name suggests, the players hold a race to the person or institution administering the estate. Upon arrival, each player can choose an order on the issues that is most favourable to him. By averaging over all possible orders of arrival, we obtain a run-to-the-bank rule. One new aspect of this rule in our context, which is not present in the standard bankruptcy context, is that a new player arriving has to take into account the effect of his choice of order on the issues on the players already present. If they stand to lose out because of this choice, the new player has to compensate them for this.

The two run-to-the-bank rules we introduce in this fashion differ in the way they treat claims within each issue. The first one (the P-rule) divides the money assigned to a particular issue proportionally, while the second one (the Q-rule) chooses an "optimistic" order on the players. The two run-to-the-bank rules turn out to be the Shapley value of the corresponding P-game and Q-game, respectively.

Finally, in section 6, we characterise both run-to-the-bank rules by means of (P- and Q-)consistency. In the context of bankruptcy games, the term consistency has been used for a number of different properties. Our definition of consistency is similar to the one used by O'Neill (1982). It is based on the idea that applying a solution concept to a particular problem and applying the same solution concept to some specific subproblems and aggregating the solutions of these subproblems should yield the same outcome. In order to properly define such a consistency property, we extend the domain of a solution concept to a wider class of problems, ie, the class of multi-issue allocation situations with awards. This extended class of situations however is not our prime interest, but its purpose is solely a technical framework in which O'Neill's characterisation can be extended in a natural way.

8.2 The model

A *multi-issue allocation situation* is a triple (N, E, C), where $N = \{1, \ldots, n\}$ is the set of players, $E \geq 0$ is the estate under contest and $C \in \mathbb{R}_+^{R \times N}$ is the matrix of claims. Every row in C represents an issue and the set of issues is denoted by $R = \{1, \ldots, r\}$. An element $c_{ki} \geq 0$ represents the amount that player $i \in N$ claims according to issue $k \in R$. If a player is not involved in a particular issue, his claim corresponding to that issue equals zero.

Every bankruptcy situation $(N, E, c) \in BR^N$ gives rise to a multi-issue allocation situation, where the issues correspond to the single-claim wills and $C \in \mathbb{R}^{N\times N}$ is the diagonal matrix with the claims c_i on the main diagonal.

With respect to the matrix of claims C, we assume the following:

- Every issue gives rise to a claim: $\sum_{i\in N} c_{ki} > 0$ for all $k \in R$.
- Every player is involved in at least one issue: $\sum_{k\in R} c_{ki} > 0$ for all $i \in N$.
- The allocation problem is nontrivial: $\sum_{k\in R}\sum_{i\in N} c_{ki} \geq E$.

For ease of notation, we define $c_{kS} = \sum_{i\in S} c_{ki}$ to be the total of claims of coalition $S \subset N$ according to issue k. Similarly, we define $c_{Ki} = \sum_{k\in K} c_{ki}$ and $c_{KS} = \sum_{k\in K}\sum_{i\in S} c_{ki}$ for all $K \subset R, S \subset N$. We denote the class of all multi-issue allocation situations with player set N by MIA^N.

As stated in the introduction, we make the basic assumption that once we are paying out money according to one particular issue, this issue must first be fully dealt with before we move on to the next. In addition, we consider two approaches on how to handle the claims within each issue. As a result, we define two *multi-issue allocation games*, a proportional game v^P based on Assumption 8.2.1 and a queue game v^Q based on Assumption 8.2.2.

Assumption 8.2.1 *If some money is allocated to the players on the basis of a particular issue, the amount of money each of the players gets is proportional to his claim according to that issue.*

In order to define the proportional game v^P, we first compute the maximum amount the players in a coalition $S \subset N$ can get when the issues are dealt with according to Assumption 8.2.1. We do this by considering all orders on the issues, so let $\tau \in \Pi(R)$. Now the players in S first address the first t issues completely, where $t = \max\{t' \mid \sum_{s=1}^{t'} c_{\tau(s),N} \leq E\}$. The part of the estate that is left, $E' = E - \sum_{s=1}^{t} c_{\tau(s),N}$, is divided proportional to the claims according to issue $\tau(t+1)$. So in total, the players in S receive[1]

[1]In the boundary case $E = c_{RN}$, we have $t = n$ and $E' = 0$ and we simply define $f_S^P(\tau) = f_S^Q(\sigma,\tau) = c_{RS}$ for all $\tau \in \Pi(R), \sigma \in \Pi(N)$

$$f_S^P(\tau) = \sum_{s=1}^{t} c_{\tau(s),S} + \frac{c_{\tau(t+1),S}}{c_{\tau(t+1),N}} E'. \tag{8.1}$$

The value of coalition $S \subset N$ is the amount of money they can guarantee themselves when the players in $N \backslash S$ are free to choose an order on the issues:

$$v^P(S) = \min_{\tau \in \Pi(R)} f_S^P(\tau).$$

Since for each $\tau \in \Pi(R)$ we have $f_S^P(\tau) + f_{N\backslash S}^P(\tau) = E$, this pessimistic point of view for S is equivalent with saying that the players in $N \backslash S$ maximise their own payoff:

$$v^P(S) = E - \max_{\tau \in \Pi(R)} f_{N\backslash S}^P(\tau). \tag{8.2}$$

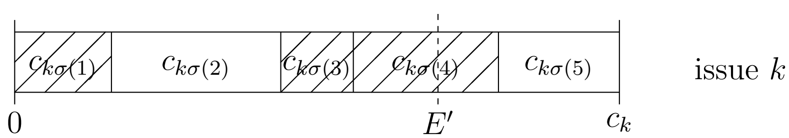
The queue game v^Q is based on Assumption 8.2.2.

Assumption 8.2.2 *If a particular coalition allocates some money to the players on the basis of a particular issue, this coalition can also decide in which order the claims corresponding to that issue are satisfied.*

To define the queue game, we first define an auxiliary function $g(S, k, \sigma, E')$, which describes how much money the players in $S \subset N$ get according to issue $k \in R$ if the order on the players is $\sigma \in \Pi(N)$ and the estate is E' with $E' < c_{kN}$. The first q players get their entire claim, where $q = \max\{q' \mid \sum_{p=1}^{q'} c_{k\sigma(p)} \leq E'\}$. The function g is then defined by

$$g(S, k, \sigma, E') = \begin{cases} E' - \displaystyle\sum_{\substack{p=1 \\ \sigma(p) \in N\backslash S}}^{q} c_{k\sigma(p)} & \text{if} \quad \sigma(q+1) \in S, \\ \displaystyle\sum_{\substack{p=1 \\ \sigma(p) \in S}}^{q} c_{k\sigma(p)} & \text{if} \quad \sigma(q+1) \notin S. \end{cases} \tag{8.3}$$

The computation of $g(S, k, \sigma, E')$ is illustrated in the following example with five players.



Coalition S consists of players $\sigma(1)$, $\sigma(3)$ and $\sigma(4)$ and corresponds to the shaded area. The estate E' is such that only the claims of the first three players can be fully satisfied ($q = 3$). Furthermore, $\sigma(q+1) \in S$, so (8.3) yields $g(S, k, \sigma, E') = E' - c_{k\sigma(2)}$, which is represented by the area to the left of E' that is not shaded, ie, the part of the estate left that is not claimed by $N \backslash S$.

Next, we compute the maximum amount the players in a coalition $S \subset N$ can get if the order on the issues is $\tau \in \Pi(R)$. As in the proportional case, the first t issues are fully dealt with and the remainder E' is distributed according to some order $\sigma \in \Pi(N)$ on the players, using Assumption 8.2.2. So, in total the players in S receive

$$f_S^Q(\sigma, \tau) = \sum_{s=1}^{t} c_{\tau(s),S} + g(S, \tau(t+1), \sigma, E'),$$

where again, $E' = E - \sum_{s=1}^{t} c_{\tau(S),N}$. The value of coalition S is then, analogous to the proportional case, given by

$$v^Q(S) = \min_{\tau \in \Pi(R)} \min_{\sigma \in \Pi(N)} f_S^Q(\sigma, \tau),$$

or equivalently, using the identity $f_S^Q(\sigma, \tau) + f_{N \backslash S}^Q(\sigma, \tau) = E$,

$$v^Q(S) = E - \max_{\tau \in \Pi(R)} \max_{\sigma \in \Pi(N)} f_{N \backslash S}^Q(\sigma, \tau). \tag{8.4}$$

Again, pessimism by the members of S boils down to letting the members of $N \backslash S$ choose an order on the issues that maximise their payoff.

It is immediately clear that the optimal order on the players that coalition $N \backslash S$ will choose puts themselves in front. So, (8.4) reduces to

$$v^Q(S) = E - \max_{\tau \in \Pi(R)} f_{N \backslash S}^Q(\tau),$$

where

$$f_{N \backslash S}^Q(\tau) = f_{N \backslash S}^Q(\hat{\sigma}, \tau) = \sum_{s=1}^{t} c_{\tau(s),N \backslash S} + \min\{c_{\tau(t+1),N \backslash S}, E'\} \tag{8.5}$$

with $\hat{\sigma} \in \Pi(N)$ such that $\hat{\sigma}^{-1}(N \backslash S) = \{1, \ldots, |N \backslash S|\}$.

8.3 Algorithms

In this section we present two algorithms to compute the proportional game v^P and the queue game v^Q corresponding to any multi-issue allocation situation (N, E, C).

8.3.1 Proportional game

Let $(N, E, C) \in MIA^N$ and let $S \subset N$ be a coalition of players. The value of S, $v^P(S)$, is computed in a number of steps:

1. Compute for every issue $k \in R$ the proportion of the total of claims corresponding to issue k that is claimed by coalition S:

   $p_k = \frac{c_{kS}}{c_{kN}}$.

2. Take $\tau \in \Pi(R)$ such that $\tau^{-1}(k) \leq \tau^{-1}(\ell)$ whenever $p_k \leq p_\ell$.

3. $v^P(S) = f_S^P(\tau)$, where $f_S^P(\tau)$ is defined in (8.1).

8.3.2 Queue game

Let $(N, E, C) \in MIA^N$ and let $S \subset N$ be a coalition of players. The value of S, $v^Q(S)$, is computed in a number of steps:

1. For all $I \subset R$ calculate

$$\begin{aligned} x_I &= \sum_{k \in I} c_{kS}, \\ y_I &= \sum_{k \in I} c_{k,N \setminus S} + \max_{k \in R \setminus I} c_{k,N \setminus S}. \end{aligned}$$

2. If $y_\emptyset \geq E$ then $v^Q(S) = 0$, otherwise proceed.

3. Find $\overline{I} \subset R$ such that

   (a) $x_{\overline{I}} + y_{\overline{I}} \geq E$,

   (b) $x_I \geq x_{\overline{I}}$ for all $I \subset R$ such that $x_I + y_I \geq E$.

   Next, find $\underline{I} \subset R$ such that

   (a) $x_{\underline{I}} + y_{\underline{I}} \leq E$,

   (b) $y_{\underline{I}} \geq y_I$ for all $I \subset R$ such that $x_I + y_I \leq E$.

4. Compute

$$v^Q(S) = \min\{x_{\overline{I}}, E - y_{\underline{I}}\}.$$

To show that this algorithm works, first of all note that it follows from the definition of f^Q that $v^Q(S)$ depends only on the aggregate claim of coalition S within each issue and not on the distribution of claims between the members of S.

The idea behind the algorithm is to represent all possible payoff profiles (x, y) *for all possible estates* by *paths* in the payoff space ($\mathbb{R}^2_+$), where x (on the horizontal axis) is the payoff to S and y (on the vertical axis) the payoff to $N \backslash S$. The aim is to find the minimum possible payoff to S given the fact that the estate equals E. The estate E is represented by the line $x + y = E$.

Coalition $N \backslash S$ has the freedom to choose an order on the issues. Now, forget the actual amount of the estate for a moment and suppose the players in $N \backslash S$ choose to address issues $I \subset R$ fully and furthermore choose one other issue in $R \backslash I$ that gives them their maximal payoff (without paying the claim of S according to that last issue). This action leads to a payoff profile of (x_I, y_I). If the estate were to equal $x_I + y_I$, the point (x_I, y_I) would represent a payoff profile which according to Assumption 8.2.2 would be feasible for $N \backslash S$ to reach.

With each order on I we associate a path connecting (x_I, y_I) to the origin. Starting with an estate of 0 (and hence, a $(0, 0)$ payoff), we are going to increase the estate to $x_I + y_I$, plotting the payoff profiles associated with all intermediate estates (determined by the order on I) in the picture. From the origin, we start paying out money to $N \backslash S$ according to the first issue in I, represented by a vertical line segment. When the estate reaches the total claim of $N \backslash S$ corresponding to the first issue, we start paying out to coalition S, represented by a horizontal line segment. After the total claim associated with the first issue has been paid out, we continue with the second issue in the order, and so on. When all issues I have been addressed, we end with a vertical line segment representing the claim of $N \backslash S$ according to the last issue. Typically, such a path looks as depicted in Figure 8.1. Note that some horizontal or vertical line segments may be absent because of zero claims.

We draw such a path for every order on I. These paths represent *all* possible payoff profiles that coalition $N \backslash S$ can reach for estates smaller than $x_I + y_I$ if they choose to address the issues in I first and put themselves in front within each issue.

Doing this for all $I \subset R$ yields *all feasible payoff profiles* (provided $N \backslash S$ acts optimally within each issue) *for any order on the issues for all estates smaller than the total of all claims.* Note that every path associated with some set $I \subsetneqq R$ of issues is part of a path connecting (x_R, y_R) to the origin.

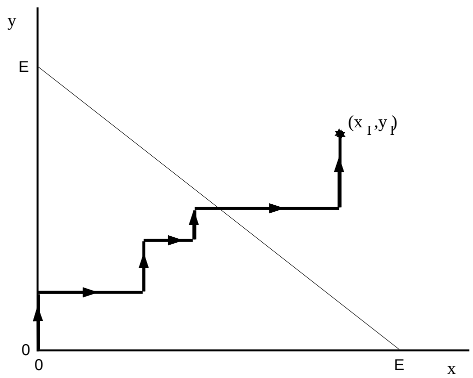


Figure 8.1

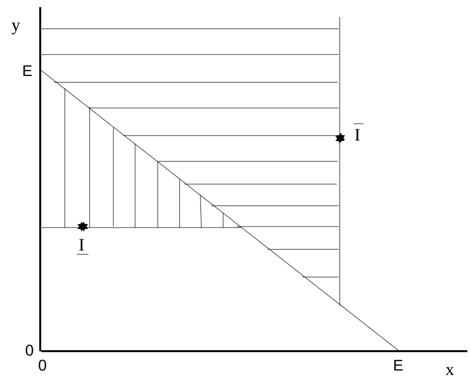


Figure 8.2

The value of coalition S is the x-coordinate of the leftmost intersection between one of these paths and the line $x + y = E$. It is immediately clear that $v^Q(S) = 0$ if $y_\emptyset \geq E$. Otherwise, take $\overline{I}$ and $\underline{I}$ as stated (which is always possible because of R and $\emptyset$, respectively).

Typically, $\overline{I}$ and $\underline{I}$ are situated as depicted in Figure 8.2. By construction, there is no $I \subset R$ giving rise to a payoff profile (x_I, y_I) in either shaded area. Note also that whereas $\overline{I}$ and $\underline{I}$ need not be uniquely determined, $x_{\overline{I}}$ and $y_{\underline{I}}$ are.

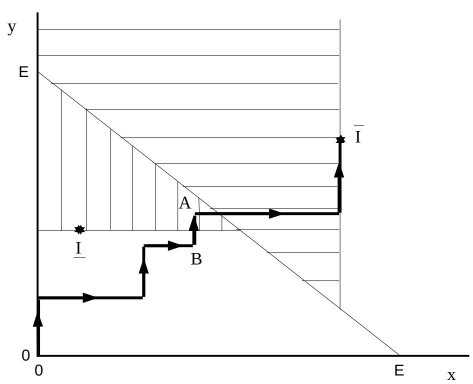


Figure 8.3

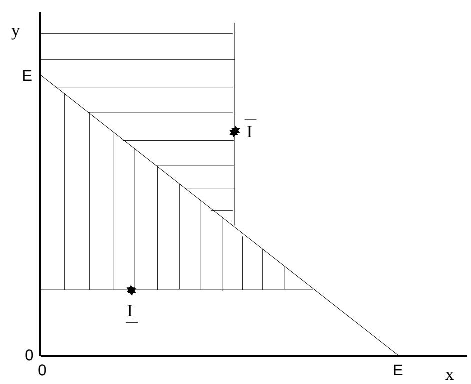


Figure 8.4

Now consider the paths associated with $\overline{I}$. We claim that there can be no path with a $\ulcorner$ kink in the shaded area. Suppose that such a path exists, as indicated in Figure 8.3, with such a kink at A. Consider all issues I^* that are fully dealt with up to point B.[2] Then by construction, (x_{I^*}, y_{I^*}) lies at or above point A. This

[2]In fact, we need the last point below A up to which all issues have been fully addressed. If A is preceded by an issue in which S claims zero, this point may be between A and B.

contradicts the fact that there is no $I \subset R$ giving rise to a payoff profile in the shaded area.

As a consequence, every path connecting $(x_{\overline{I}}, y_{\overline{I}})$ to the origin must cross the line $x + y = E$ to the right of $(E - y_{\underline{I}}, y_{\underline{I}})$ if $x_{\overline{I}} + y_{\underline{I}} > E$ (the case depicted in Figure 8.3). The same holds for every path connecting any point above the line $x + y = E$ to the origin. Hence, $v^Q(S) \geq E - y_{\underline{I}}$. Furthermore, there is a path going through $(E - y_{\underline{I}}, y_{\underline{I}})$, because $N \backslash S$ can guarantee themselves $y_{\underline{I}}$ by addressing issues $\underline{I}$ first. Therefore, $v^Q(S) = E - y_{\underline{I}}$ if $x_{\overline{I}} + y_{\underline{I}} > E$.

Similarly, if $x_{\overline{I}} + y_{\underline{I}} < E$, as depicted in Figure 8.4, every path intersecting the line $x + y = E$ must do so to the right of $(x_{\overline{I}}, E - x_{\overline{I}})$ and there is a path going through this point. Hence, in this case $v^Q(S) = x_{\overline{I}}$.

If $x_{\overline{I}} + y_{\underline{I}} = E$, both sets of arguments can be used. One should also note that all these arguments still hold in case $(x_{\underline{I}}, y_{\underline{I}})$ or $(x_{\overline{I}}, y_{\overline{I}})$ lie on the line $x + y = E$ rather than below or above.

Summarising these cases, we obtain

$$v^Q(S) = \min\{x_{\overline{I}}, E - y_{\underline{I}}\},$$

as stated in the algorithm.

8.4 Properties of multi-issue allocation games

In this section we look at some of the properties that multi-issue allocation games of both types possess. First, we prove that the worth of a coalition in the queue game is smaller than the worth of that coalition in the corresponding proportional game. This means that the queue approach is more pessimistic than the proportional approach.

Proposition 8.4.1 *Let $(N, E, C) \in MIA^N$ be a multi-issue allocation situation with corresponding games v^P and v^Q. Then $v^Q(S) \leq v^P(S)$ for all $S \subset N$.*

Proof: Let $S \subset N$ and let $\tau^\circ \in \Pi(R)$ be an ordering on the issues where the maximum in (8.2) is obtained. For any order $\tau \in \Pi(R)$, $\min\{c_{\tau(t+1),N\backslash S}, E'\}$ in (8.5) exceeds $\frac{c_{\tau(t+1),N\backslash S}}{c_{\tau(t+1),N}} E'$ in (8.1). So, in particular, this is the case for τ°. But then certainly, $\max_{\tau \in \Pi(R)} f^Q_{N\backslash S}(\tau) \geq f^P_{N\backslash S}(\tau^\circ)$ and hence, $v^Q(S) \leq v^P(S)$. $\square$

As was the case for bankruptcy games with references in the previous chapter, multi-issue allocation games turn out to be exact.

Theorem 8.4.2 *Let $(N, E, C) \in MIA^N$. Then both corresponding games v^P and v^Q are exact.*

Proof: Let $S \subset N$ and let $\tau^\circ \in \Pi(R)$ and $\sigma^\circ \in \Pi(N)$ be such that $f_S^Q(\sigma^\circ, \tau^\circ)$ is minimal. Define $x = (f_i^Q(\sigma^\circ, \tau^\circ))_{i \in N}$. Then $\sum_{i \in N} x_i = E = v^Q(N)$ and $\sum_{i \in T} x_i = f_T^Q(\sigma^\circ, \tau^\circ) \geq \min_{\tau \in \Pi(R)} \min_{\sigma \in \Pi(N)} f_T^Q(\sigma, \tau) = v^Q(T)$ for all coalitions $T \subset N$. So, $x \in C(v^Q)$. Furthermore, $\sum_{i \in S} x_i = f_S^Q(\sigma^\circ, \tau^\circ) = v^Q(S)$. Hence, v^Q is exact. The proof for v^P is similar. □

In the proof of Theorem 8.4.2 we showed that $(f_i^Q(\sigma^\circ, \tau^\circ))_{i \in N}$ is a core element of the queue game v^Q for certain $\sigma^\circ \in \Pi(N)$ and $\tau^\circ \in \Pi(R)$. This property can be extended to all orders on the issues, so for all $\sigma \in \Pi(N), \tau \in \Pi(R)$ we have

$$(f_i^Q(\sigma, \tau))_{i \in N} \in C(v^Q)$$

and similarly for the proportional game, for all $\tau \in \Pi(R)$,

$$(f_i^P(\tau))_{i \in N} \in C(v^P).$$

Theorem 8.4.3 *Let $v \in TU^N$ be a nonnegative exact game. Then there exists a multi-issue allocation situation $(N, E, C) \in MIA^N$ such that both corresponding games v^P and v^Q equal v.*

Proof: If $|N| = 1$, the result is obvious. Otherwise, define $E = v(N)$ and take for all $S \subsetneq N, S \neq \emptyset$ an $x^S \in C(v)$ such that $\sum_{i \in S} x_i^S = v(S)$. Interpret these core elements as issues and gather them (as rows) in the $(2^n - 2) \times n$ claim matrix C. Because $c_{kN} = E$ for all $k \in R$, no issue is addressed partially and v^P and v^Q coincide.
Now, let $S \subset N$. By construction, there is a row $k' \in R$ such that $c_{k'S} = v(S)$ and because all issues are core elements of v, $c_{kS} \geq v(S)$ for all $k \in R$. Hence, $v^P(S) = \min_{\tau \in \Pi(R)} f_S^P(\tau) = \min_{k \in R} c_{kS} = v(S)$. Therefore, v, v^P and v^Q coincide. □

From Theorems 8.4.2 and 8.4.3 we conclude that the class of multi-issue allocation games coincides with the class of nonnegative exact games. Because not every four-player exact game is convex, it follows from Theorem 8.4.3 that multi-issue allocation games with more than three players need not be convex.

A well known property of a convex game is that its Shapley value belongs to the core. Rabie (1981) shows that this does not hold in general for exact games. However, Theorem 8.4.4 shows that the Shapley value of a nonnegative exact game belongs to the core cover.

Theorem 8.4.4 *Let $v \in TU^N$ be a nonnegative exact game. Then $\Phi(v) \in CC(v)$.*

Proof: First, use Theorem 8.4.3 to construct a multi-issue allocation situation $(N, E, C) \in MIA^N$ such that $v^P = v$. Next, let $i \in N$. Then supperadditivity implies $v^P(S) - v^P(S\backslash\{i\}) \geq v^P(\{i\}) = m_i(v^P)$ for all $S \subset N$ such that $i \in S$. Furthermore,

$$\begin{aligned}
v^P(S) - v^P(S\backslash\{i\}) &= \left[E - \max_{\tau\in\Pi(R)} f^P_{N\backslash S}(\tau)\right] - \left[E - \max_{\tau\in\Pi(R)} f^P_{(N\backslash S)\cup\{i\}}(\tau)\right] \\
&= \max_{\tau\in\Pi(R)} f^P_{(N\backslash S)\cup\{i\}}(\tau) - \max_{\tau\in\Pi(R)} f^P_{N\backslash S}(\tau) \\
&\leq \max_{\tau\in\Pi(R)} f^P_{N\backslash S}(\tau) + \max_{\tau\in\Pi(R)} f^P_{\{i\}}(\tau) - \max_{\tau\in\Pi(R)} f^P_{N\backslash S}(\tau) \\
&= E - \left[E - \max_{\tau\in\Pi(R)} f^P_{\{i\}}(\tau)\right] \\
&= v^P(N) - v^P(N\backslash\{i\}) \\
&= M_i(v^P).
\end{aligned}$$

Hence, the marginal contribution of i to every coalition $S : i \in S$ is bounded by $m_i(v^P)$ and $M_i(v^P)$. Because the Shapley value is the average of these marginal contributions, $\Phi(v^P) \in CC(v^P)$ and hence, $\Phi(v) \in CC(v)$. □

Sprumont (1990) shows that every convex game has a pmas (see section 4.2). This does not hold for exact games, as is shown by the following example.

Example 8.4.1 Consider the multi-issue allocation situation with player set $N = \{1, \ldots, 4\}$, estate $E = 22$ and claim matrix

$$C = \begin{bmatrix} 6 & 6 & 5 & 3 \\ 12 & 0 & 2 & 6 \end{bmatrix}.$$

The corresponding queue game is as follows:

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$v^Q(S)$	6	0	2	3	12	11	9	2	6	8	14	15	16	8	22

To show that v^Q has no pmas, suppose $(x^S)_{S\subset N, S\neq\emptyset}$ satisfies (4.1) and (4.2). Then we subsequently have:

- $v^Q(\{1,3\}) = 11$ and $v^Q(\{1,3,4\}) = 16$ imply $x_4^{\{1,3,4\}} \leq 16 - 11 = 5$;
- $x_4^{\{1,3,4\}} \leq 5$ implies $x_4^{\{3,4\}} \leq 5$;
- $x_4^{\{3,4\}} \leq 5$ and $v^Q(\{3,4\}) = 8$ imply $x_3^{\{3,4\}} \geq 3$;
- $x_3^{\{3,4\}} \geq 3$ and $v^Q(\{2,4\}) = 6$ imply $\sum_{i\in\{2,3,4\}} x_i^{\{2,3,4\}} \geq 9$.

The last statement contradicts (4.1) and hence, the exact game v^Q possesses no pmas. $\triangleleft$

8.5 The run-to-the-bank rule

A multi-issue allocation rule is a function $\Psi : MIA^N \to \mathbb{R}^N$ assigning to every multi-issue allocation situation $(N,E,C) \in MIA^N$ a vector $\Psi(N,E,C) \in \mathbb{R}^N$ such that $\sum_{i\in N} \Psi_i(N,E,C) = E$ (efficiency) and $0 \leq \Psi_i(N,E,C) \leq c_{Ri}$ for all $i \in N$ (reasonability). We define two rules, called run-to-the-bank rules, based on Assumptions 8.2.1 and 8.2.2. These rules are based on the run-to-the-bank rule for bankruptcy situations (see section 7.2). The *proportional run-to-the-bank rule* is defined as

$$\rho^P = \frac{1}{|N|!} \sum_{\sigma\in\Pi(N)} \rho^P(\sigma),$$

where for all $\sigma \in \Pi(N)$, $\rho^P(\sigma) \in \mathbb{R}^N$ is defined recursively by

$$\rho^P_{\sigma(p)}(\sigma) = \max_{\tau\in\Pi(R)} \left[f^P_{\sigma(p)}(\tau) - \sum_{q=1}^{p-1} \left(\rho^P_{\sigma(q)}(\sigma) - f^P_{\sigma(q)}(\tau) \right) \right] \tag{8.6}$$

for all $p \in \{1,\ldots,n\}$. The vector $\rho^P(\sigma)$ is interpreted as follows. To divide the estate, a “race” is held between the players and they arrive at the person or institution administering the estate in the order given by σ. The first player that arrives,

$\sigma(1)$, can choose the order in which the issues are dealt with and receives his payoff accordingly. Of course, he will choose that order $\tau \in \Pi(R)$ for which his payoff $f^P_{\sigma(1)}(\tau)$ is maximal. Next, player $\sigma(2)$ arrives and he is asked to do the same. However, if he chooses an order different from the first one, he has to compensate player $\sigma(1)$ for the difference between his settled payoff $\rho^P_{\sigma(1)}(\sigma)$ and his payoff according to the new order. Taking this into account, the second player will pick that order that maximises his own payoff minus the corresponding compensation payments. The same procedure is applied to all subsequent players, each having to compensate all his predecessors.

The *queue run-to-the-bank rule* is defined as

$$\rho^Q = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} \rho^Q(\sigma),$$

where for all $\sigma \in \Pi(N)$, $\rho^Q(\sigma) \in \mathbb{R}^N$ is defined recursively by

$$\rho^Q_{\sigma(p)}(\sigma) = \max_{\tau \in \Pi(R)} \max_{\gamma \in \Pi(N)} \left[f^Q_{\sigma(p)}(\gamma, \tau) - \sum_{q=1}^{p-1} \left(\rho^Q_{\sigma(q)}(\sigma) - f^Q_{\sigma(q)}(\gamma, \tau) \right) \right] \tag{8.7}$$

for all $p \in \{1, \ldots, n\}$. The interpretation is similar to the proportional case. The only difference is that the queue payoff function f^Q is used rather than the proportional function f^P and that in accordance with Assumption 8.2.2, players *also* have to specify an order γ on the players. It is immediately clear that it is optimal for player $\sigma(p)$, who arrives at the administrator at position p, to choose γ in such a way that he himself and all preceding players, $\sigma(1), \ldots, \sigma(p-1)$, whom he has to compensate, are in front of the queue. This can be done by setting $\gamma = \sigma$.

Proposition 8.5.1 *In (8.7), taking $\gamma = \sigma$ is optimal.*

As a result of Proposition 8.5.1, (8.7) can be rewritten as

$$\rho^Q_{\sigma(p)}(\sigma) = \max_{\tau \in \Pi(R)} \left[f^Q_{\sigma(p)}(\sigma, \tau) - \sum_{q=1}^{p-1} \left(\rho^Q_{\sigma(q)}(\sigma) - f^Q_{\sigma(q)}(\sigma, \tau) \right) \right] \tag{8.8}$$

In order to prove that both run-to-the-bank rules equal the Shapley values of their respective corresponding games, we first relate them to the marginal vectors. For this, we define for any order $\sigma \in \Pi(N)$ the reverse order $\sigma^* \in \Pi(N)$ by $\sigma^*(p) = \sigma(n-p+1)$ for all $p \in \{1, \ldots, n\}$.

Lemma 8.5.2 *Let $(N, E, C) \in MIA^N$ be a multi-issue allocation situation with corresponding games v^P and v^Q. Then $\rho^P(\sigma) = m^{\sigma^*}(v^P)$ and $\rho^Q(\sigma) = m^{\sigma^*}(v^Q)$ for all $\sigma \in \Pi(N)$.*

Proof: We only prove the statement for the queue game; the proof for the proportional game is similar. Let $\sigma \in \Pi(N)$. Then for all $p \in \{1, \dots n\}$ we have, using (8.8),

$$\begin{aligned}
\rho^Q_{\sigma(p)}(\sigma) &= \max_{\tau\in\Pi(R)} \left[f^Q_{\sigma(p)}(\sigma,\tau) - \sum_{q=1}^{p-1} \left(\rho^Q_{\sigma(q)}(\sigma) - f^Q_{\sigma(q)}(\sigma,\tau) \right) \right] \\
&= \max_{\tau\in\Pi(R)} f^Q_{\{\sigma(1),\dots,\sigma(p)\}}(\sigma,\tau) - \sum_{q=1}^{p-1} \rho^Q_{\sigma(q)}(\sigma) \\
&= \max_{\tau\in\Pi(R)} f^Q_{\{\sigma(1),\dots,\sigma(p)\}}(\sigma,\tau) - \max_{\tau\in\Pi(R)} f^Q_{\{\sigma(1),\dots,\sigma(p-1)\}}(\sigma,\tau) \\
&= E - \min_{\tau\in\Pi(R)} f^Q_{\{\sigma(p+1),\dots,\sigma(n)\}}(\sigma,\tau) - E + \min_{\tau\in\Pi(R)} f^Q_{\{\sigma(p),\dots,\sigma(n)\}}(\sigma,\tau) \\
&= -\min_{\tau\in\Pi(R)} f^Q_{\{\sigma^*(1),\dots,\sigma^*(n-p)\}}(\sigma,\tau) + \min_{\tau\in\Pi(R)} f^Q_{\{\sigma^*(1),\dots,\sigma^*(n-p+1)\}}(\sigma,\tau) \\
&= -\min_{\tau\in\Pi(R)} \min_{\bar\sigma\in\Pi(N)} f^Q_{\{\sigma^*(1),\dots,\sigma^*(n-p)\}}(\bar\sigma,\tau) + \\
&\qquad \min_{\tau\in\Pi(R)} \min_{\bar\sigma\in\Pi(N)} f^Q_{\{\sigma^*(1),\dots,\sigma^*(n-p+1)\}}(\bar\sigma,\tau) \\
&= -v^Q(\{\sigma^*(1),\dots,\sigma^*(n-p)\}) + v^Q(\{\sigma^*(1),\dots,\sigma^*(n-p+1)\}) \\
&= m^{\sigma^*}_{\sigma^*(n-p+1)}(v^Q) \\
&= m^{\sigma^*}_{\sigma(p)}(v^Q),
\end{aligned}$$

where the third equality follows from recursively substituting the formulas for $\rho^Q_{\sigma(q)}(\sigma)$. □

Theorem 8.5.3 *Let $(N, E, C) \in MIA^N$ be a multi-issue allocation situation with corresponding games v^P and v^Q. Then $\rho^P = \Phi(v^P)$ and $\rho^Q = \Phi(v^Q)$.*

Proof: This result follows immediately from Lemma 8.5.2 and from the observation that $\{\sigma^* \mid \sigma \in \Pi(N)\} = \Pi(N)$. □

8.6 Consistency

O'Neill (1982) characterised his recursive completion method (run-to-the-bank rule) for bankruptcy situations by means of the property of *consistency*. A bankruptcy rule f is called consistent if for every bankruptcy situation (N, E, c) the following two procedures yield the same outcome:

1. Apply f to the whole bankruptcy situation (N, E, c).

2. For each player $j \in N$, consider the subsituation where this player j receives his claim (truncated to the estate), $\min\{c_j, E\}$, and the other players $N\backslash\{j\}$ divide the remainder of the estate $E - \min\{c_j, E\}$ among themselves using the original claims $c_{N\backslash\{j\}}$ and the rule f. The solution of the original situation (N, E, c) is then the average of the solutions of these n subsituations.

So, a bankruptcy rule f is consistent if for each bankruptcy situation (N, E, c) and each player $i \in N$ the following equality holds:

$$f_i(N, E, c) = \frac{1}{n}\left(\min\{c_i, E\} + \sum_{j\in N\backslash\{i\}} f_i(N\backslash\{j\}, E - \min\{c_j, E\}, c_{N\backslash\{j\}})\right),$$

where the first term on the right hand side represents the payoff to player i if he receives his truncated claim and the other terms correspond to the subsituations in which the other players play this role.

In this section we generalise this result by O'Neill and characterise the proportional and queue run-to-the-bank rules by means of consistency. Contrary to the standard bankruptcy framework, however, we cannot simply give a player his claim and send him away. Not only is it unclear what he should actually receive, but more fundamentally, by omitting him from the situation, vital information on the interdependency between the issues is lost.

To solve this, we extend our framework and broaden the domain of these rules to a larger class of situations, namely multi-issue allocation situations *with awards*. The idea behind this construction is that instead of sending a player away, we keep him in and fix the payoff that he eventually receives (the award). We should stress, that although this new class has a nice interpretation in itself, it is not directly intended as an extension of multi-issue allocation situations, but as a (technical) framework in which the characterisation of the run-to-the-bank rule by O'Neill can be generalised in a natural way.

A *multi-issue allocation situation with awards* is a 4-tuple (N, E, C, μ), where $\mu \in \mathbb{R}^F$ represents some award vector to some specific coalition $F \subset N$, which has already been agreed upon. The sum of these awards cannot exceed the estate, so $\sum_{i \in F} \mu_i \leq E$. Furthermore, $\sum_{i \in F} \mu_i = E$ if $F = N$. Note that a multi-issue allocation situation without awards is a special case with $F = \emptyset$.

A rule Ψ is a function assigning to every multi-issue allocation situation with awards (N, E, C, μ) a vector $\Psi(N, E, C, \mu) \in \mathbb{R}^N$ such that $\sum_{i \in N} \Psi_i(N, E, C, \mu) = E$ and $\Psi_F(N, E, C, \mu) = \mu$. That is, for a rule in this environment it should hold that every player in F gets exactly his award. Note that contrary to the situation without awards, we do not impose reasonability[3] on Ψ. On this new class of situations we also define two run-to-the-bank rules. For this, we first fix an order on the players in F, so let $\gamma \in \Pi(F)$. The proportional run-to-the-bank rule with awards is defined as:

$$\rho^P(\mu) = \frac{1}{|N \backslash F|!} \sum_{\sigma \in \Pi^\gamma(N)} \rho^P(\sigma, \mu),$$

where

$$\Pi^\gamma(N) = \left\{ \sigma \in \Pi(N) \,|\, \forall_{q \in \{1, \ldots, |F|\}} : \sigma(q) = \gamma(q) \right\}$$

and for all $\sigma \in \Pi^\gamma(N)$, $\rho^P(\sigma, \mu) \in \mathbb{R}^N$ is defined recursively by

$$\rho^P_{\sigma(p)}(\sigma, \mu) = \mu_{\sigma(p)}$$

for all $p \in \{1, \ldots, n\}$ such that $\sigma(p) \in F$ and

$$\rho^P_{\sigma(p)}(\sigma, \mu) = \max_{\tau \in \Pi(R)} \left\{ f^P_{\sigma(p)}(\tau) - \sum_{q=1}^{p-1} \left[\rho^P_{\sigma(q)}(\sigma, \mu) - f^P_{\sigma(q)}(\tau) \right] \right\}$$

for all $p \in \{1, \ldots, n\}$ such that $\sigma(p) \notin F$.

Note that the run-to-the-bank rule does not depend on the actual choice of γ. This definition differs from the run-to-the-bank rule without awards (8.6) in two respects: every player $i \in F$ gets μ_i rather than the maximum expression in (8.6) and the players in F have to be compensated (which is accomplished in an order

[3]To guarantee reasonability of the run-to-the-bank rules with awards as defined below, we would have to make some unnecessary diverting assumptions. We just note that for the specific multi-issue allocation situations with awards that are derived from a standard multi-issue allocation situation using either run-to-the-bank rule, reasonability is satisfied.

$\sigma \in \Pi^\gamma(N)$ by putting them at the front). Note that for $F = \emptyset$, the two definitions coincide.

In a similar fashion, we define the queue run-to-the-bank rule with awards:

$$\rho^Q(\mu) = \frac{1}{|N \backslash F|!} \sum_{\sigma \in \Pi^\gamma(N)} \rho^Q(\sigma, \mu),$$

where for all $\sigma \in \Pi^\gamma(N)$, $\rho^Q(\sigma, \mu) \in \mathbb{R}^N$ is defined recursively by

$$\rho^Q_{\sigma(p)}(\sigma, \mu) = \mu_{\sigma(p)}$$

for all $p \in \{1, \ldots, n\}$ such that $\sigma(p) \in F$ and

$$\rho^Q_{\sigma(p)}(\sigma, \mu) = \max_{\tau \in \Pi(R)} \left\{ f^Q_{\sigma(p)}(\sigma, \tau) - \sum_{q=1}^{p-1} \left[\rho^Q_{\sigma(q)}(\sigma, \mu) - f^Q_{\sigma(q)}(\sigma, \tau) \right] \right\}$$

for all $p \in \{1, \ldots, n\}$ such that $\sigma(p) \notin F$. Note that this definition generalises (8.8) rather than (8.7). Proposition 8.5.1 can easily be extended to the situation with awards, so letting each player choose an order on the players would result in an equivalent definition.

For all $i \in N \backslash F$ and $\tau \in \Pi(R)$ we define the remainder functions

$$r^P_i(\tau) = f^P_{F \cup \{i\}}(\tau) - \sum_{j \in F} \mu_j \qquad \left(= f^P_i(\tau) + \sum_{j \in F} [f^P_j(\tau) - \mu_j]\right)$$

and

$$r^Q_i(\tau) = f^Q_{F \cup \{i\}}(\sigma, \tau) - \sum_{j \in F} \mu_j,$$

where $\sigma \in \Pi^\gamma(N)$ is such that $\sigma(|F| + 1) = i$. These remainder functions represent the amount of money player i gets according to order τ, when he has to ensure that every player $j \in F$ gets μ_j. A rule Ψ is called *P-consistent* if for all multi-issue allocation situations with awards (N, E, C, μ) and all $i \in N \backslash F$ we have

$$\Psi_i(N, E, C, \mu) = \frac{1}{|N \backslash F|} \left(\max_{\tau \in \Pi(R)} r^P_i(\tau) + \sum_{\substack{j \in N \backslash F \\ j \neq i}} \Psi_i(N, E, C, \mu^j) \right), \tag{8.9}$$

where $\mu^j \in \mathbb{R}^{F \cup \{j\}}$ is such that $\mu^j_F = \mu$ and $\mu^j_j = \max_{\tau \in \Pi(R)} r^P_j(\tau)$. Ψ is *Q-consistent* if for all (N, E, C, μ) and all $i \in N \backslash F$

$$\Psi_i(N,E,C,\mu) = \frac{1}{|N\backslash F|}\left(\max_{\tau\in\Pi(R)} r_i^Q(\tau) + \sum_{\substack{j\,\in\, N\backslash F\\ j\neq i}} \Psi_i(N,E,C,\mu^j)\right)$$

with $\mu_j^j = \max_{\tau\in\Pi(R)} r_j^Q(\tau)$. The idea behind consistency in this context is as follows. Let i be a player in $N\backslash F$. Then the first term between parentheses is the amount of money player i gets when he maximises his own payoff by choosing an order on the issues, keeping in mind the players in F have to receive their awards. Next, let $j \in N\backslash F, j \neq i$. Now suppose that player j receives his maximal remainder. Then a new situation arises where player j has been awarded some fixed amount. The amount of money player i receives in this new situation is given by applying rule Ψ on the old μ extended with the fixed award to player j. A rule is called consistent if applying it directly yields the same outcome as averaging over all $|N\backslash F|$ situations where one of the non-fixed player get their maximum.

Theorem 8.6.1 *The proportional run-to-the-bank rule ρ^P is the unique P-consistent rule and the queue run-to-the-bank rule ρ^Q is the unique Q-consistent rule.*

Proof: We only give the proof for ρ^P. The proof for ρ^Q goes along similar lines. First, we prove that ρ^P satisfies P-consistency. Let $i \in N\backslash F$. Then

$$\begin{aligned}
&\rho_i^P(\mu) = \\
&= \frac{1}{|N\backslash F|!}\sum_{\sigma\in\Pi^\gamma(N)} \rho_i^P(\sigma,\mu) \\
&= \frac{1}{|N\backslash F|!}\sum_{j\in N\backslash F}\sum_{\sigma\in\Pi^{\gamma,j}(N)} \rho_i^P(\sigma,\mu) \\
&= \frac{1}{|N\backslash F|!}\sum_{\sigma\in\Pi^{\gamma,i}(N)} \max_{\tau\in\Pi(R)}\left\{f_i^P(\tau) - \sum_{q=1}^{\sigma^{-1}(i)-1}\left[\rho_{\sigma(q)}^P(\sigma,\mu) - f_{\sigma(q)}^P(\tau)\right]\right\} + \\
&\quad \frac{1}{|N\backslash F|!}\sum_{\substack{j\,\in\, N\backslash F\\ j\neq i}}\sum_{\sigma\in\Pi^{\gamma,j}(N)} \max_{\tau\in\Pi(R)}\left\{f_i^P(\tau) - \sum_{q=1}^{\sigma^{-1}(i)-1}\left[\rho_{\sigma(q)}^P(\sigma,\mu) - f_{\sigma(q)}^P(\tau)\right]\right\} \\
&= \frac{1}{|N\backslash F|!}\sum_{\sigma\in\Pi^{\gamma,i}(N)} \max_{\tau\in\Pi(R)}\left\{f_{F\cup\{i\}}^P(\tau) - \sum_{j\in F}\mu_j\right\} +
\end{aligned}$$

$$
\begin{aligned}
& \frac{1}{|N\backslash F|!} \sum_{\substack{j \in N\backslash F \\ j \neq i}} \sum_{\sigma \in \Pi^{\gamma,j}(N)} \max_{\tau \in \Pi(R)} \left\{ f_i^P(\tau) - \sum_{q=1}^{\sigma^{-1}(i)-1} \left[\rho_{\sigma(q)}^P(\sigma, \mu^j) - f_{\sigma(q)}^P(\tau) \right] \right\} \\
= \; & \frac{1}{|N\backslash F|!} (|N\backslash F| - 1)! \max_{\tau \in \Pi(R)} r_i^P(\tau) + \\
& \frac{1}{|N\backslash F|} \sum_{\substack{j \in N\backslash F \\ j \neq i}} \frac{1}{(|N\backslash F| - 1)!} \sum_{\sigma \in \Pi^{\gamma,j}(N)} \rho_i^P(\mu^j) \\
= \; & \frac{1}{|N\backslash F|} \left(\max_{\tau \in \Pi(R)} r_i^P(\tau) + \sum_{\substack{j \in N\backslash F \\ j \neq i}} \rho_i^P(\mu^j) \right),
\end{aligned}
$$

where $\Pi^{\gamma,j}(N) = \{\sigma \in \Pi^{\gamma}(N) \mid \sigma(|F|+1) = j\}$ for $j \in N\backslash F$.

Uniqueness of the P-consistent rule is proved by induction on the size of F. Assume that rule Ψ is P-consistent. For $F = N$, $\Psi(N, E, C, \mu) = \mu$ by definition. Next, (8.9) completely determines the solutions of all situations with $|F| = |N| - 1$. Repeating this procedure until $F = \emptyset$, we conclude that there is a unique P-consistent rule, which is the proportional run-to-the-bank rule. □

Chapter 9

A composite MIA approach

9.1 Introduction

In the previous chapter, we extended the bankruptcy model to encompass situations in which the agents can have multiple claims on the estate, each as a result of a particular issue. For such multi-issue allocation (MIA) situations we proposed an extension of the run-to-the-bank rule[1] as solution for this new class of problems. As is the case for the original rule, this extended run-to-the-bank rule turns out to coincide with the Shapley value of the corresponding multi-issue allocation game.

Contrary to bankruptcy games, however, multi-issue allocation games need not be convex. Consequently, there exist multi-issue allocation situations for which the run-to-the-bank solution is not a core element of the corresponding game. In this chapter, which is based on González-Alcón et al. (2003), we extend the run-to-the-bank rule in a different way, such that it always yields a core element.

Instead of considering the issues and the players combined, as in Chapter 8, in this chapter we propose a two-stage extension, called the composite run-to-the-bank rule. First, we explicitly allocate the estate to the issues (according to a marginal vector), and then, within each issue the money is divided among the agents using the standard run-to-the-bank rule. An alternative view on composite solutions is given in Casas-Méndez et al. (2002).

Based on Aumann and Maschler (1985), we define the concept of (self-)duality for multi-issue allocation situations and show that both the queue run-to-the-bank-rule and the composite run-to-the-bank rule are self-dual. We characterise the composite extension by means of the property of issue-consistency, which like P- and

[1] In this chapter, we only refer to the *queue* approach of the previous chapter.

Q-consistency in section 8.6 generalises the consistency property that was first used by O'Neill (1982).

This chapter is organised as follows. In section 2, we define the composite extension of the run-to-the-bank rule and show that this rule always yields a core element. In section 3, we define self-duality and prove that both extensions of the run-to-the-bank rule satisfy this property. Finally, in section 4, we characterise the composite run-to-the-bank rule by means of issue-consistency and we show that this rule is estate monotonic.

9.2 The composite run-to-the-bank rule

In this section, we extend the run-to-the-bank rule for bankruptcy situations to the class of MIA situations. Contrary to the extension in section 8.5, the present extension, which we call the *composite run-to-the-bank rule*, involves multiple runs to the bank, once by the issues and within each issue by the players.

Throughout this chapter, we denote the bankruptcy game corresponding to the situation $(R, E, (c_{kN})_{k \in R})$ by $v^R_{E,C}$. We denote the vector $(c_{ki})_{i \in N}$ for $k \in R$ by C_k.

As stated in section 7.2, the run-to-the-bank rule for bankruptcy games, RTB, coincides with the Shapley value of the corresponding bankruptcy game and can thus be expressed as

$$RTB(N, E, c) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(v_{E,c}).$$

Let $(N, E, C) \in MIA^N$. For $\tau \in \Pi(R)$ and $\sigma \in \Pi(N)$, we define the *composite marginal vector* as[2]

$$mm^{\tau,\sigma}(N, E, C) = \sum_{k \in R} m^{\sigma^*}(v_{x_k, C_k}),$$

where $x = m^{\tau^*}(v^R_{E,C})$. The following lemma follows from Lemma 8.5.2.

Lemma 9.2.1 *Let $(N, E, c) \in BR^N$ and $\sigma \in \Pi(N)$. Then*

$$m^\sigma(v_{E,c}) = \rho(\sigma^*).$$

[2]Formally, in a bankruptcy situation all claims are positive (see section 7.2). As a result, (N, x_k, C_k) need not be a proper bankruptcy situation. In the analysis of this chapter, we can ignore this and we can apply (7.1) to obtain a bankruptcy game.

As a result of this lemma, the composite marginal vector $mm^{\tau,\sigma}$ can be viewed as a race to the bank, where the issues arrive in order τ and the players in order σ.

The set of all composite marginal vectors is a subset of the core of the corresponding MIA queue game, as is shown in the following proposition.

Proposition 9.2.2 *Let* $(N, E, C) \in MIA^N$. *Then*

$$mm^{\tau,\sigma}(N, E, C) \in C(v^Q)$$

for all $\tau \in \Pi(R)$, $\sigma \in \Pi(N)$.

Proof: Let $\tau \in \Pi(R), \sigma \in \Pi(N)$ and let $z = mm^{\tau,\sigma}(N, E, C)$. Let $x = m^{\tau^*}(v^R_{E,C})$ and $t = \max\{t \mid \sum_{p=1}^{t} c_{\tau(p),N} \leq E\}$. With x_k as estate for issue $k \in R$, we have a collection of bankruptcy situations $\{(N, x_k, C_k)\}_{k \in R}$. However, at most one of them is a nontrivial situation: in the situations $\tau(1), \ldots, \tau(t)$ the estate equals the sum of all the claims and in the situations $\tau(t+2), \ldots, \tau(r)$ the estate equals zero. Let y be the marginal vector corresponding to σ of the only possible nontrivial bankruptcy situation $(N, x_{\tau(t+1)}, C_{\tau(t+1)})$:

$$y = m^{\sigma^*}(v_{x_{\tau(t+1)}, C_{\tau(t+1)}}).$$

We can express vector z as

$$z = y + \sum_{p=1}^{t} C_{\tau(p)}.$$

Let $S \subset N$. Then with $E' = \sum_{i \in N} y_i$, we have

$$\begin{aligned} \sum_{i \in S} z_i &= g(S, \tau(t+1), \sigma, E') + \sum_{p=1}^{t} c_{\tau(p),S} \\ &= f^Q_S(\sigma, \tau) \geq v^Q(S). \end{aligned}$$

Hence, $z \in C(v^Q)$. □

A general relation of inclusion between the set of marginal vectors of the queue game and the set of composite marginal vectors cannot be established, as is shown in the following example.

Example 9.2.1 Let $(N, E, C) \in MIA^N$ with $N = \{1, 2, 3\}$, $E = 10$ and

$$C = \begin{bmatrix} 9 & 5 & 0 \\ 3 & 7 & 7 \end{bmatrix}.$$

The queue game associated with this situation is

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v^Q(S)$	0	0	0	3	3	1	10

The sets of marginal and composite marginal vectors can be easily calculated. The results are given in the following table.

$\sigma \in \Pi(N)$	m^σ	$\tau \in \Pi(R)$	$\sigma \in \Pi(N)$	$mm^{\tau,\sigma}$
123	$(0, 3, 7)$	12	$123, 132, 312$	$(9, 1, 0)$
132	$(0, 7, 3)$		$213, 231, 321$	$(5, 5, 0)$
213	$(3, 0, 7)$	21	$123, 213$	$(3, 7, 0)$
231	$(9, 0, 1)$		$132, 312$	$(3, 0, 7)$
312	$(3, 7, 0)$		231	$(0, 7, 3)$
321	$(9, 1, 0)$		321	$(0, 3, 7)$

The table shows that $m^{231}(v^Q)$ is not a composite marginal vector and that $mm^{12,213}(N, E, C)$ does not belong to the set of marginal vectors of the game v^Q. $\triangleleft$

We define the *composite run-to-the-bank rule*, $mRTB$, by

$$mRTB(N, E, C) = \frac{1}{r!} \sum_{\tau \in \Pi(R)} \sum_{k \in R} RTB(N, m_k^\tau(v_{E,C}^R), C_k) \tag{9.1}$$

for all $(N, E, C) \in MIA^N$. The $mRTB$ rule can be interpreted as the result of two races: first, the issues "run to the bank" for the money, and next, there are r races among the players within each issue. As is the case for the RTB rule for bankruptcy situations, the claims are satisfied as much as possible by the order of arrival.

The $mRTB$ rule first takes the marginal vectors of the "issue game" $v_{E,C}^R$. Associated with each marginal vector $m^\tau(v_{E,C}^R)$ we have r bankruptcy games whose estates are given by the components of the marginal vector. Next, we take for each player the sum of the RTB solutions of these r situations. Finally, the average among all the marginals is computed. It is readily seen that the $mRTB$ rule can be expressed as

$$mRTB(N, E, C) = \frac{1}{r!} \sum_{\tau \in \Pi(R)} \frac{1}{n!} \sum_{\sigma \in \Pi(N)} mm^{\tau,\sigma}(N, E, C). \tag{9.2}$$

If we start with a bankruptcy situation $(N, E, c) \in BR^N$ and construct the corresponding MIA situation (N, E, C) with the claims on the main diagonal of C, then $RTB(N, E, c) = mRTB(N, E, C)$. So, the composite run-to-the-bank rule is indeed an extension of the run-to-the-bank rule. However, $mRTB$ does not in general coincide with the Shapley value of the game. In fact, the $mRTB$ rule is not even game-theoretic, ie, two situations leading to the same game might yield different outcomes.

The composite run-to-the-bank rule provides a way of obtaining an element of the core of the corresponding queue game without calculating the characteristic function. This is stated in the following theorem.

Theorem 9.2.3 *Let* $(N, E, C) \in MIA^N$. *Then*

$$mRTB(N, E, C) \in C(v^Q).$$

Proof: In Proposition 9.2.2, we showed that every composite marginal vector lies in the core. The $mRTB$ outcome, being the average of these composite marginals vectors according to equation (9.2), then also is an element of the core, which is a convex set. □

As an alternative to the $mRTB$ rule, another way to extend the RTB rule in a two-stage way would be to apply the RTB rule twice: $\sum_{k \in R} RTB(N, x_k, C_k)$ with $x = RTB(R, E, (c_{kN})_{k \in R}) = \Phi(v^R_{E,C})$. However, this solution can lie outside the core of the corresponding queue game, as the next example shows.

Example 9.2.2 Consider the MIA situation $(N, E, C) \in MIA^N$ with $N = \{1, 2, 3\}$, $E = 51$ and

$$C = \begin{bmatrix} 0 & 2 & 6 \\ 0 & 1 & 24 \\ 24 & 2 & 0 \end{bmatrix}.$$

The queue game associated with this situation is

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v^Q(S)$	16	3	22	21	46	27	51

We have $x = RTB(R, E, (c_{kN})_{k \in R}) = (\frac{16}{3}, \frac{67}{3}, \frac{70}{3})$ and $\sum_{k \in R} RTB(N, x_k, C_k) = (\frac{67}{3}, \frac{5}{2}, \frac{157}{6})$. As $\frac{5}{2} < 3 = v^Q(\{2\})$, this solution is not in the core of v^Q. ◁

9.3 Self-duality

For a MIA situation (N, E, C) we define $D(S) = c_{RS}$, ie, the total claim of the players in coalition S, and we define $D = D(N)$. Recall that we assume $D \geq E$. To distinguish between the various games, we denote the queue game corresponding to (N, E, C) by $v^Q_{E,C}$.

The proof of the following lemma is partly taken from Wintein (2002).

Lemma 9.3.1 *Let $(N, E, C) \in MIA^N$. Then for all $S \subset N$,*

$$v^Q_{E,C}(S) = v^Q_{D-E,C}(N \backslash S) + D(S) - D + E.$$

Proof: Let $S \subset N$. To calculate the value of $v^Q_{E,C}(S)$, we must find an ordering on the players $\sigma \in \Pi(N)$ and an ordering on the issues $\tau \in \Pi(R)$ such that the total amount assigned to coalition S, $f^Q_S(\sigma, \tau)$, is minimal. Obviously, σ can be any ordering in which the players in S are at the end.

In Figure 9.1 we represent all the claims of matrix C in the order indicated by τ and σ, ie, $c_{\tau(1)\sigma(1)}, c_{\tau(1)\sigma(2)}, \ldots, c_{\tau(r)\sigma(n)}$. The claims associated with players in S are shaded. The total claim is divided into two parts of lengths E and $D - E$, as the figure shows. From the way in which σ and τ are chosen, the dark zone in the E part is as small as possible, and it is precisely $v^Q_{E,C}(S)$.

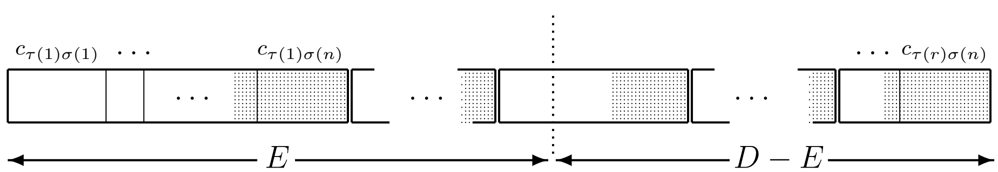


Figure 9.1: Proof of Lemma 9.3.1

If we consider now the MIA situation $(N, D - E, C)$ and we want to calculate $v^Q_{D-E,C}(N \backslash S)$, we must find $\sigma' \in \Pi(N)$ and $\tau' \in \Pi(R)$ such that the white zone in the $D - E$ segment is minimised. The length of this zone is indeed $v^Q_{D-E,C}(N \backslash S)$. Since this is in a sense the complementary problem of the first one, this minimum is reached for σ^* and τ^*, ie, the reverse orderings of σ and τ.

On the other hand, we have that the E segment is the sum of its white and shaded parts. The white part within E will be the total white zone $D(N \backslash S)$ minus the white zone in the $D-E$ segment. The shaded part of E is $v^Q_{E,C}(S)$, as was indicated above. So,

$$E = v^Q_{E,C}(S) + D(N\backslash S) - v^Q_{D-E,C}(N\backslash S).$$

From the equality $D = D(S) + D(N\backslash S)$, we conclude that the statement holds. □

The next lemma gives us the relation between the marginal vectors of the two queue games with estates E and $D-E$.

Lemma 9.3.2 *Let $(N, E, C) \in MIA^N$. Then*

$$m^\sigma(v^Q_{E,C}) = (c_{Ri})_{i\in N} - m^{\sigma^*}(v^Q_{D-E,C})$$

for all $\sigma \in \Pi(N)$.

Proof: Let $\sigma \in \Pi(N)$ and $p \in \{1,\dots,n\}$. Let $i = \sigma(p)$ and let S be the coalition $\{\sigma(1),\dots,\sigma(p-1)\}$. Then

$$m^\sigma_i(v^Q_{E,C}) = v^Q_{E,C}(S\cup\{i\}) - v^Q_{E,C}(S).$$

From Lemma 9.3.1 we then have

$$\begin{aligned} m^\sigma_i(v^Q_{E,C}) &= v^Q_{D-E,C}(N\backslash(S\cup\{i\})) + D(S\cup\{i\}) - D + E \\ &\quad -[v^Q_{D-E,C}(N\backslash S) + D(S) - D + E] \\ &= D(\{i\}) + v^Q_{D-E,C}(N\backslash(S\cup\{i\})) - v^Q_{D-E,C}(N\backslash S) \\ &= D(\{i\}) - m^{\sigma^*}_i(v^Q_{D-E,C}). \end{aligned}$$

From $D(\{i\}) = c_{Ri}$ the result follows. □

Following Aumann and Maschler (1985), given a rule f we can define its *dual* f^* by using f to share not the estate E but the gap $D-E$. So, each player receives his claim (the part he would receive if the estate were big enough) minus the corresponding part of the losses:

$$f^*(N, E, C) = (c_{Ri})_{i\in N} - f(N, D-E, C).$$

A rule is called *self-dual* if $f^* = f$. We show that both ρ^Q (defined by (8.8)) and $mRTB$ are self-dual.

Proposition 9.3.3 *The ρ^Q rule is self-dual.*

Proof: Let $(N, E, C) \in MIA^N$. Since ρ^Q coincides with the Shapley value of the associated queue game, we have

$$\rho^Q(N, E, C) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(v^Q_{E,C}).$$

From Lemma 9.3.2 it then follows that

$$\begin{aligned}
\frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(v^Q_{E,C}) &= \frac{1}{n!} \sum_{\sigma \in \Pi(N)} [(c_{Ri})_{i \in N} - m^{\sigma^*}(v^Q_{D-E,C})] \\
&= (c_{Ri})_{i \in N} - \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(v^Q_{D-E,C}) \\
&= (c_{Ri})_{i \in N} - \rho^Q(N, D - E, C).
\end{aligned}$$

This shows that ρ^Q is self-dual. □

As a result of the previous proposition, the RTB rule is self-dual for bankruptcy situations as well, which was first proved by Curiel (1988).

Proposition 9.3.4 *The $mRTB$ rule is self-dual.*

Proof: Let $(N, E, C) \in MIA^N$. We denote by v^R_E and v^R_{D-E} the characteristic functions of the games induced by the bankruptcy situations $(R, E, (c_{kN})_{k \in R})$ and $(R, D - E, (c_{kN})_{k \in R})$, respectively. Then,

$$\begin{aligned}
mRTB(N, E, C) &= \frac{1}{r!} \sum_{\tau \in \Pi(R)} \sum_{k \in R} RTB(N, m^\tau_k(v^R_E), C_k) \\
&= \frac{1}{r!} \sum_{\tau \in \Pi(R)} \sum_{k \in R} RTB(N, c_{kN} - m^{\tau^*}_k(v^R_{D-E}), C_k) \\
&= \frac{1}{r!} \sum_{\tau \in \Pi(R)} \sum_{k \in R} \left[C_k - RTB(N, c_{kN} - c_{kN} + m^{\tau^*}_k(v^R_{D-E}), C_k)\right] \\
&= \sum_{k \in R} C_k - \frac{1}{r!} \sum_{\tau \in \Pi(R)} \sum_{k \in R} RTB(N, m^\tau_k(v^R_{D-E}), C_k) \\
&= (c_{Ri})_{i \in N} - mRTB(N, D - E, C),
\end{aligned}$$

where for the second equality we use Lemma 9.3.1 and for the third equality we use self-duality of the RTB rule for bankruptcy situations. Hence, the $mRTB$ rule is self-dual. □

9.4 Issue consistency and monotonicity

In this section we characterise the composite run-to-the-bank rule as a consistent extension of the RTB rule for bankruptcy situations to multi-issue allocations situations. This so-called issue-consistency allows us to easily establish estate monotonicity of the $mRTB$ rule. See section 8.6 for a wider discussion on consistency.

A MIA rule f is called *issue-consistent* if for each MIA situation $(N, E, C) \in MIA^N$ the following holds:

$$\begin{aligned} f(N,E,C) &= \frac{1}{r}\sum_{k\in R}[f(N,\min\{E,c_{kN}\},C_k) \\ &\quad + f(N,\max\{E-c_{kN},0\},C_{-k})], \end{aligned} \tag{9.3}$$

where C_{-k} is the claim matrix C from which issue k has been deleted.

The first term of the summation in (9.3) applies the rule f to a one-issue allocation situation (so basically a bankruptcy situation), while the second term applies f to a MIA situation with $r-1$ issues. So, successively applying this property allows us to extend any bankruptcy rule to the class of multi-issue allocation situations. Analogous to the characterisation in section 8.6, every bankruptcy rule has a unique issue-consistent extension.

Theorem 9.4.1 *The $mRTB$ rule is the unique issue-consistent extension of the RTB rule.*

Proof: Let $(N, E, C) \in MIA^N$. Then

$$\begin{aligned} & mRTB(N,E,C) = \\ &= \frac{1}{r!}\sum_{\tau\in\Pi(R)}\sum_{k\in R} RTB(N, m_k^\tau(v_{E,C}^R), C_k) \\ &= \frac{1}{r!}\sum_{k\in R}\sum_{\ell\in R}\sum_{\substack{\tau\in\Pi(R)\\ \tau(r)=\ell}} RTB(N, m_k^\tau(v_{E,C}^R), C_k) \\ &= \frac{1}{r!}\sum_{k\in R}\sum_{\substack{\tau\in\Pi(R)\\ \tau(r)=k}} RTB(N, m_k^\tau(v_{E,C}^R), C_k) + \\ &\quad \frac{1}{r!}\sum_{k\in R}\sum_{\ell\in R\setminus\{k\}}\sum_{\substack{\tau\in\Pi(R)\\ \tau(r)=k}} RTB(N, m_\ell^\tau(v_{E,C}^R), C_\ell) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{r!}(r-1)!\sum_{k\in R} RTB(N,\min\{c_{kN},E\},C_k) + \\
&\quad \frac{1}{r!}(r-1)!\sum_{k\in R}\frac{1}{(r-1)!}\sum_{\tau\in\Pi(N\backslash\{k\})}\sum_{\ell\in R\backslash\{k\}} RTB(N,m_\ell^\tau(v^{R\backslash\{k\}}),C_\ell) \\
&= \frac{1}{r}\sum_{k\in R} mRTB(N,\min\{c_{kN},E\},C_k) + \\
&\quad \frac{1}{r}\sum_{k\in R} mRTB(N,\max\{E-c_{kN},0\},C_{-k}),
\end{aligned}
$$

where $v^{R\backslash\{k\}}$ is the bankruptcy game associated with $(R\backslash\{k\},\max\{E - c_{kN},0\},(c_{\ell N})_{\ell\in R\backslash\{k\}})$. Hence, the $mRTB$ rule is issue-consistent. Uniqueness follows from a similar reasoning as in the proof of Theorem 8.6.1. □

Issue-consistency allows us to show that the composite run-to-the-bank rule is estate monotonic. A MIA rule f is *estate monotonic* if for every pair of MIA situations (N,E,C) and (N,E',C) with $E'\geq E$ we have

$$f_i(N,E',C)\geq f_i(N,E,C)$$

for all $i\in N$.

Theorem 9.4.2 *The $mRTB$ rule is estate monotonic.*

Proof: We show that the $mRTB$ rule is estate monotonic by induction on the number of issues r. If $r=1$ then $mRTB$ coincides with RTB and this rule is estate monotonic on the class of banckruptcy games (cf. Curiel (1988)).
Next, assume that $mRTB$ is estate monotonic for situations with $r-1$ issues. Let C be a claim matrix with r rows. By issue-consistency we have

$$
\begin{aligned}
mRTB(N,E,C) &= \frac{1}{r}\sum_{k\in R}[mRTB(N,\min\{E,c_k\},C_k) \\
&\quad +mRTB(N,\max\{E-c_k,0\},C_{-k})].
\end{aligned}
$$

In the first term inside the brackets we basically apply the RTB rule to a bankruptcy situation. So, by estate monotonicity of the RTB rule, this term increases if the estate is raised. The second term is the application of $mRTB$ to a $(r-1)$-issue allocation situation, which by the induction hypothesis satisfies the estate monotonicity property. Adding up all terms, we have that $mRTB$ is estate monotonic. □

Chapter 10

Bankruptcy with a priori unions

10.1 Introduction

In many situations in which agents interact, they do so in groups. Cooperative game theory studies such situations by taking into account what each particular coalition of players can achieve on its own. These values of the coalitions are subsequently taken into account in determining a fair allocation of the value of the grand coalition. Often, however, some coalitions play a special role, in that they arise in a natural way from the underlying situation. If these naturally arising groups form a partition of the grand coalition, they are usually referred to as *a priori unions* (cf. Owen (1975)).

One interesting class of problems in which the role of a priori unions has been studied is the class of bankruptcy situations (see section 7.2). In a bankruptcy situation, there is an estate to be divided among a number of players, whose total claim exceeds the estate available. In many situations, these players can be divided into a priori unions, based on the nature or cause of their claims. Eg, when a firm goes bankrupt, the creditors can usually be grouped in a natural way by distinguishing between claims on the basis of outstanding bonds, equity or commercial transactions.

The main focus in the bankruptcy literature is on finding rules assigning to each bankruptcy situation an allocation of the estate, which satisfy some appealing properties. One natural way to analyse the class of bankruptcy situations with a priori unions is to extend well-known standard bankruptcy rules to this class. Eg, Casas-Méndez et al. (2003) extend the adjusted proportional rule by considering a two-stage procedure in which the estate is first divided among the unions, and subsequently the amount that each union receives is divided among its members.

In this chapter, which is based on Casas-Méndez et al. (2002), we present two extensions of the constrained equal award (CEA) rule. The first extension involves a similar two-stage procedure as in Casas-Méndez et al. (2003). We relate this extension to the CEA solution of a corresponding TU game with a priori unions, which is inspired by Owen (1977). We provide two characterisations of this two-stage extension, inspired by previous results by Dagan (1996) and Herrero and Villar (2002). The second extension of the CEA rule is based on the random arrival rule introduced in O'Neill (1982) and it is characterised by a consistency property.

The outline of this chapter is as follows. In section 2, we formally define the class of bankruptcy situations with a priori unions and some related concepts that are used throughout this chapter. In section 3, the problem of extending standard bankruptcy rules is addressed and the first extension is presented. In section 4, we provide the two characterisations of the two-stage extension of the CEA rule. Section 5 contains the second extension and deals with the concept of consistency.

10.2 Bankruptcy with a priori unions

A *bankruptcy situation with a priori unions* is a 4-tuple $(N, E, c, \mathcal{P})$ where (N, E, c) is a standard bankruptcy problem and $\mathcal{P} = \{P_k\}_{k \in R}$ is a partition of the set of players into unions, R being the set of unions. We denote by BU^N the set of all bankruptcy problems with a priori unions with player set N.

A *bankruptcy with a priori unions rule* is a function $\varphi : BU^N \to \mathbb{R}^N$ that assign to each $(N, E, c, \mathcal{P}) \in BU^N$ a payoff vector $\varphi(N, E, c, \mathcal{P}) \in \mathbb{R}^N$ such that for all $i \in N$, $0 \leq \varphi_i(N, E, c, \mathcal{P}) \leq c_i$ and $\sum_{i \in N} \varphi_i(N, E, c, \mathcal{P}) = E$.

For $(N, E, c, \mathcal{P}) \in BU^N$ we define the corresponding bankruptcy situation among the unions $(R, E, c^{\mathcal{P}}) \in BR^N$, the so-called *quotient problem*, where $c^{\mathcal{P}} = (c_k^{\mathcal{P}})_{k \in R}$ is the vector of total claims of the unions, so $c_k^{\mathcal{P}} = \sum_{i \in P_k} c_i$ for each union P_k of players. Note that $(R, E, c^{\mathcal{P}})$ is a well defined bankruptcy problem.

A *cooperative game with transferable utility with a priori unions* is a triple $(N, v, \mathcal{P})$ where (N, v) is a TU game and $\mathcal{P} = \{P_k\}_{k \in R}$ is a partition of the set of players. For $(N, v, \mathcal{P})$, we define the corresponding TU game among the unions $(R, v^{\mathcal{P}})$, the quotient game, by $v^{\mathcal{P}}(L) = v(\cup_{k \in L} P_k)$ for all $L \subset R$.

A bankruptcy situation with a priori unions gives rise in a natural way to a multi-issue allocation situation, where the issues correspond to the unions. In order to analyse such situations, in Chapter 8 we define two corresponding games, the proportional game and the queue game. In this chapter, we consider a variation on the former: instead of dividing the estate proportional to the claims within the final issue to be handled, we apply an arbitrary bankruptcy rule f to this problem. Note that for all f, the resulting game is exact (which follows from the proof of Theorem 8.4.2), but not necessarily convex.

The link with multi-issue allocation situations and corresponding games is illustrated in the following example, where for the definition of the CEA rule we refer to section 7.2.

Example 10.2.1 Consider the 4-player bankruptcy problem $(N, E, c) \in BR^N$ with $E = 10$ and $c = (6, 2, 8, 5)$. Suppose players 1 and 2 form a union and players 3 and 4 another one, that is, $\mathcal{P} = \{\{1, 2\}, \{3, 4\}\}$.

This situation gives rise to the 4-player multi-issue allocation situation $(N, E, C) \in MIA^N$ with $E = 10$ and

$$C = \begin{bmatrix} 6 & 2 & 0 & 0 \\ 0 & 0 & 8 & 5 \end{bmatrix}.$$

Take $S = \{1, 3\}$. In order to determine $v^{CEA}(S)$, we first compute, for both $\tau \in \Pi(R)$, $f_S^{CEA}(\tau)$, the quantity that coalition S receives if the issues are handled in order τ and the final issue is resolved using CEA:

τ	$f_S^{CEA}(\tau)$
$(1, 2)$	$6 + CEA_3(\{3, 4\}, 2, (8, 5)) = 7$
$(2, 1)$	$CEA_3(\{3, 4\}, 10, (8, 5)) = 5$

So, $v^{CEA}(S) = \min_{\tau \in \Pi(R)} f_S^{CEA}(\tau) = 5$. Similarly, taking $T = \{1, 4\}$, we obtain $v^{CEA}(T) = 5$, $v^{CEA}(S \cup T) = 8$ and $v^{CEA}(S \cap T) = 0$. Hence, $v^{CEA}(S) + v^{CEA}(T) > v^{CEA}(S \cup T) + v^{CEA}(S \cap T)$. So, although v^{CEA} is exact, it is not convex. $\triangleleft$

10.3 Extending bankruptcy rules: a two-step procedure

In this section we consider a way to extend a game-theoretic bankruptcy rule to a rule for bankruptcy situations with a priori unions. We use the CEA rule to illustrate this extension. We also connect our CEA solution for a bankruptcy situation with a priori unions to the corresponding TU game with a priori unions.

In order to divide the total estate among the players, one approach is first to divide the estate among the unions and second to divide the amount of each union among the players in this union. Let $f : BR^N \to \mathbb{R}^N$ be a game-theoretic bankruptcy rule. We define the two-stage extension $\bar{f} : BU^N \to \mathbb{R}^N$ as follows. Let $(N, E, c, \mathcal{P}) \in BU^N$ be a bankruptcy problem with a priori unions. Define $E_k^f = f_k(R, E, c^{\mathcal{P}})$ for all $k \in R$ and next, for all $i \in P_k, k \in R$, define $\bar{f}_i(N, E, c, \mathcal{P}) = f_i(P_k, E_k^f, (c_j)_{j \in P_k})$.

The $\overline{CEA}$ rule for bankruptcy situations with a priori unions generalises the standard CEA rule for bankruptcy situations, in the sense that both $\overline{CEA}(N, E, c, \mathcal{P}^N)$ and $\overline{CEA}(N, E, c, \mathcal{P}^n)$ coincide with $CEA(N, E, c)$, where $\mathcal{P}^n$ is the discrete partition $\mathcal{P}^n = \{\{1\}, \ldots, \{n\}\}$ and $\mathcal{P}^N$ is the trivial partition $\mathcal{P}^N = \{N\}$. Also note that by construction, $\overline{CEA}_k(R, E, c^{\mathcal{P}}, \mathcal{P}^R) = E_k^{CEA}$ for all $k \in R$. Since $(R, E, c^{\mathcal{P}}, \mathcal{P}^R)$ is basically indistinguishable from $(R, E, c^{\mathcal{P}})$, we refer to both situations as the quotient problem associated with $(N, E, c, \mathcal{P})$.

The $\overline{CEA}$ solution of a bankruptcy situation with a priori unions coincides with the CEA solution for the corresponding TU game with a priori unions, which we are going to define next.

First, recall that the *utopia vector* of a game $v \in TU^N$, $M(v)$, is defined by $M_i(v) = v(N) - v(N \backslash \{i\})$ for all $i \in N$. This vector is used to define the CEA *solution* of the game, which is defined for all $i \in N$ by $CEA_i(N, v) = \min\{\lambda, M_i(v)\}$, where λ is such that $\sum_{i \in N} \min\{\lambda, M_i(v)\} = v(N)$.[1] This solution divides the worth of the total coalition, $v(N)$, among the players in such a way that all of them obtain the same amount with the restriction that no player can get more than his utopia payoff. Note that for $(N, E, c) \in BR^N$, we have $CEA(N, E, c) = CEA(v_{E,c})$.

Now, let $(N, v, \mathcal{P})$ be a TU game with a priori unions. The constrained equal award solution of this game, $CEA(N, v, \mathcal{P})$ is defined in two steps. First, the payoff to each union $P_k \in \mathcal{P}$ equals $CEA_k(R, v^{\mathcal{P}})$, ie, the constrained equal award solution of the quotient game. In the second step, the payoff to each union is divided among its players. To do this, we consider for every player $i \in N$ his cooperation possibilities with the players outside his union. A similar idea is used in Owen (1977), where a

[1]The CEA rule for TU games is only well-defined for a subclass of such games. If the game is exact, then the CEA rule is well-defined. The same holds for the CEA rule for games with a priori unions, which we define later on, where exactness of the underlying game is sufficient.

modification of the Shapley value for TU games with a priori unions is defined. Let $P_k \in \mathcal{P}$ and let $i \in P_k$. The "claim" of player i is defined as his contribution to the coalition $\cup_{\ell \in R\setminus\{k\}} P_\ell \cup \{i\}$, that is, $M_i(v,\mathcal{P}) = v(\cup_{\ell \in R\setminus\{k\}} P_\ell \cup \{i\}) - v(\cup_{\ell \in R\setminus\{k\}} P_\ell)$. The constrained equal award solution of the game $(N,v,\mathcal{P})$ for player $i \in P_k, k \in R$ is then defined by

$$CEA_i(N,v,\mathcal{P}) = CEA_i(P_k, CEA_k(R, v^{\mathcal{P}}), (M_j(v,\mathcal{P}))_{j\in P_k}).$$

The $\overline{CEA}$ solution of a bankruptcy situation with a priori unions coincides with the CEA solution of the corresponding game $(N, v^{CEA}, \mathcal{P})$, as is shown in the following proposition, where v^{CEA} is the multi-issue allocation game obtained by applying the CEA rule in the last issue.

Proposition 10.3.1 *For every $(N,E,c,\mathcal{P}) \in BU^N$ we have that $\overline{CEA}(N,E,c,\mathcal{P}) = CEA(N, v^{CEA}, \mathcal{P})$.*

Proof: Let $(N,E,c,\mathcal{P}) \in BU^N$. First, it follows of the definition of the game v^{CEA} (or indeed of v^f for any bankruptcy rule f) that

$$v^{CEA}(\cup_{k\in L} P_k) = \max\{E - \sum_{i \in N\setminus \cup_{k\in L} P_k} c_i, 0\}$$

for all $L \subset R$ and hence, the games $(R, (v^{CEA})^{\mathcal{P}})$ and $(R, v_{E,c^{\mathcal{P}}})$ coincide. So,

$$CEA_k(R, (v^{CEA})^{\mathcal{P}}) = CEA_k(R, v_{E,c^{\mathcal{P}}}) = CEA_k(R, E, c^{\mathcal{P}}) = E_k^{CEA}$$

for all $k \in R$.

Next, for $i \in P_k$,

$$M_i(v^{CEA},\mathcal{P}) = \begin{cases} CEA_i(P_k, E, (c_j)_{j\in P_k}) & \text{if } E \leq c_k^{\mathcal{P}}, \\ c_i & \text{if } E > c_k^{\mathcal{P}}, \end{cases}$$

where for both $\cup_{\ell\in R\setminus\{k\}} P_\ell \cup \{i\}$ and $\cup_{\ell\in R\setminus\{k\}} P_\ell$ any worst order on the issues starts with issue k.

From the previous, we have

$$\begin{aligned} \overline{CEA}_i(N,E,c,\mathcal{P}) &= CEA_i(P_k, E_k^{CEA}, (c_j)_{j\in P_k}) \\ &= CEA_i(P_k, E_k^{CEA}, (M_j(v^{CEA},\mathcal{P}))_{j\in P_k}) \\ &= CEA_i(P_k, CEA_k(R,(v^{CEA})^{\mathcal{P}}), (M_j(v^{CEA},\mathcal{P}))_{j\in P_k}) \\ &= CEA_i(N, v^{CEA}, \mathcal{P}) \end{aligned}$$

for all $i \in P_k$, where for the second equality, observe that truncating i's claim to $CEA_i(P_k, E, (c_j)_{j \in P_k})$ in case $E \leq c_k^{\mathcal{P}}$ does not affect his payoff. This concludes the proof. $\square$

Using the associated bankruptcy game rather than the CEA multi-issue allocation game with a priori unions yields a different outcome, as is illustrated in the following example.

Example 10.3.1 Consider the 3-player bankruptcy situation $(N, E, c) \in BR^N$ with $E = 400$ and $c = (100, 100, 400)$. Suppose player 1 forms a union and players 2 and 3 form another one, that is, $\mathcal{P} = \{P_1, P_2\}$ with $P_1 = \{1\}$ and $P_2 = \{2, 3\}$. To find the $\overline{CEA}$ solution of the bankruptcy situation with unions $(N, E, c, \mathcal{P})$, we first consider the bankruptcy situation $(R, E, c^{\mathcal{P}})$ among the unions. We obtain $CEA(R, E, c^{\mathcal{P}}) = (100, 300)$ and then $\overline{CEA}(N, E, c, \mathcal{P}) = (100, 100, 200)$. By Proposition 10.3.1, we have that $CEA(N, v^{CEA}, \mathcal{P}) = (100, 100, 200)$. To find the CEA solution of the game $(N, v_{E,c}, \mathcal{P})$ we first consider the corresponding game among the unions $(R, v_{E,c}^{\mathcal{P}})$, which yields $CEA(R, v_{E,c}^{\mathcal{P}}) = (100, 300)$. Hence, $CEA_1(N, v_{E,c}, \mathcal{P}) = 100$. To determine the allocation of $CEA_{23}(R, v_{E,c}^{\mathcal{P}})$ to players 2 and 3, we compute the utopia payoffs $M_2(v_{E,c}, \mathcal{P}) = 0$ and $M_3(v_{E,c}, \mathcal{P}) = 300$. Hence $CEA(N, v_{E,c}, \mathcal{P}) = (100, 0, 300) \neq \overline{CEA}(N, E, c, \mathcal{P})$. $\triangleleft$

10.4 Characterisations of $\overline{CEA}$

In this section we use the axiomatic method to support the two-stage procedure presented in the previous section. We provide two different sets of axioms to characterise the $\overline{CEA}$ rule, extending two previous characterisations of the CEA rule for standard bankruptcy problems. Consider the following properties for a rule $\varphi : BU^N \to \mathbb{R}^N$.

Composition (COMP): For each $(N, E, c, \mathcal{P}) \in BU^N$, $\varphi(N, E, c, \mathcal{P}) = \varphi(N, E', c, \mathcal{P}) + \varphi(N, E - E', c - \varphi(N, E', c, \mathcal{P}), \mathcal{P})$ for all $0 \leq E' \leq E$.

This property considers the situation in which after the estate (E') has been divided among the players, this estate is reevaluated and turns out to be a bigger amount (E). In these cases, we have two options. We can cancel the initial division and apply the rule to the new problem, or we can preserve the initial division and apply the rule to the increment of the estate by considering a new vector of claims, taking into account the quantities already received. The composition property says that both options should lead to the same result.

Path independence (PI): For each $(N, E, c, \mathcal{P}) \in BU^N$, $\varphi(N, E, c, \mathcal{P}) = \varphi(N, E, \varphi(N, E', c, \mathcal{P}), \mathcal{P})$ for all $E' \geq E$.

Here, the opposite situation is considered, one where the estate (E) is smaller than the one initially considered (E'). Then, we can apply the rule to the new problem or divide the new estate by taking the initial divisions as claim vector. Path independence states that both ways of proceeding should result in the same payoffs.

Equal treatment within the unions (ET): For each $(N, E, c, \mathcal{P}) \in BU^N$ and for each pair of players i, j within a union $P_k \in \mathcal{P}$ such that $c_i = c_j$, $\varphi_i(N, E, c, \mathcal{P}) = \varphi_j(N, E, c, \mathcal{P})$.

This property requires that players of the same union with equal claims obtain equal payoffs.

Quotient problem property (QPP): For each $(N, E, c, \mathcal{P}) \in BU^N$ and for each union $P_k \in \mathcal{P}$, $\sum_{i \in P_k} \varphi_i(N, E, c, \mathcal{P}) = \varphi_k(R, E, c^{\mathcal{P}}, \mathcal{P}^R)$.

In a bankruptcy situation with unions we can consider the associated quotient problem where the unions negotiate about the division of the estate. After this, a negotiation within every union takes place. The quotient problem property states that the total payoff to the players of a union in the initial problem must equal the payoff to this union in the quotient problem. Note that if φ is the two-step extension $\bar{f}$ of a bankruptcy rule f, then $\varphi_k(R, E, c^{\mathcal{P}}, \mathcal{P}^R) = E_k^f = f_k(R, E, c^{\mathcal{P}})$.

Invariance under claims truncation within the unions (ICT): For each $(N, E, c, \mathcal{P}) \in BU^N$ and for every player i in a union $P_k \in \mathcal{P}$ such that $c_i > \sum_{j \in P_k} \varphi_j(N, E, c, \mathcal{P})$, we have $\varphi(N, E, c, \mathcal{P}) = \varphi(N, E, c', \mathcal{P})$, where $c'_j = c_j$ for all $j \in N \setminus \{i\}$ and $c'_i = \sum_{j \in P_k} \varphi_j(N, E, c, \mathcal{P})$.

Suppose the claim of a player is greater than the total quantity that his union gets according to φ. Then ICT states that the outcome is not affected if we replace the claim of this player by the total payoff to his union.

Sustainability of players within the unions (SUS): For each $(N, E, c, \mathcal{P}) \in BU^N$ and for every player i who is *sustainable* within his union $P_k \in \mathcal{P}$, ie, $\sum_{j \in P_k} \min\{c_i, c_j\} \leq \varphi_k(R, E, c^{\mathcal{P}}, \mathcal{P}^R)$, we have $\varphi_i(N, E, c, \mathcal{P}) = c_i$.

This property establishes a protective criterion within each union in the sense that small claims should be completely satisfied. The claim of player i is considered sustainable within his union if the worth of this union in the quotient problem is enough to pay each player in this union his claim, truncated to the claim of player i.

Composition and path independence are in essence identical to the corresponding properties for bankruptcy rules (cf. Young (1988) and Moulin (1987)). Equal treatment within the unions is a weak version of the equal treatment property for bankruptcy rules. Invariance under claims truncation within the unions and sustainability of players within the unions are natural extensions of the corresponding properties for bankruptcy rules to this context of a priori unions. Note that the quotient problem property implies that the rule φ must involve some two-step procedure to obtain the solution.

In the following theorem we present the first characterisation of the $\overline{CEA}$ rule. This theorem is inspired by a similar result for the CEA rule for bankruptcy situations in Dagan (1996).

Theorem 10.4.1 *The $\overline{CEA}$ rule is the unique rule for bankruptcy situations with a priori unions that satisfies equal treatment within the unions, composition, the quotient problem property and invariance under claims truncation within the unions.*

Proof: First, we show that $\overline{CEA}$ satisfies these four properties. Equal treatment within the unions and the quotient problem property follow immediately from the definitions. To show that $\overline{CEA}$ satisfies the composition property, let $P_k \in \mathcal{P}$ and let $i \in P_k$.

Then

$$\overline{CEA}_i(N, E, c, \mathcal{P}) = CEA_i(P_k, E_k^{CEA}, (c_j)_{j\in P_k}).$$

Consider now $0 \leq E' \leq E$. Then

$$\overline{CEA}_i(N, E', c, \mathcal{P}) = CEA_i(P_k, E_k^{CEA'}, (c_j)_{j\in P_k}),$$

with $E_k^{CEA'} = CEA_k(R, E', c^{\mathcal{P}}) \leq E_k^{CEA}$. Define $c' = c - \overline{CEA}(N, E', c, \mathcal{P})$. Then we have

$$\overline{CEA}_i(N, E - E', c', \mathcal{P}) = CEA_i(P_k, CEA_k(R, E - E', (c')^{\mathcal{P}}), (c'_j)_{j\in P_k}).$$

Because the constrained equal award rule for bankruptcy situations satisfies composition (cf. Dagan (1996)), we have

$$\begin{aligned} E_k^{CEA} - E_k^{CEA'} &= CEA_k(R, E, c^{\mathcal{P}}) - CEA_k(R, E', c^{\mathcal{P}}) \\ &= CEA_k(R, E - E', c^{\mathcal{P}} - CEA(R, E', c^{\mathcal{P}})) \\ &= CEA_k(R, E - E', (c')^{\mathcal{P}}), \end{aligned}$$

where in the last equality we use QPP. From the previous, it follows that

$$\overline{CEA}_i(N, E, c, \mathcal{P}) =$$

$$\begin{aligned} &= CEA_i(P_k, E_k^{CEA}, (c_j)_{j\in P_k}) \\ &= CEA_i(P_k, E_k^{CEA'}, (c_j)_{j\in P_k}) + CEA_i(P_k, E_k^{CEA} - E_k^{CEA'}, (c'_j)_{j\in P_k}) \\ &= \overline{CEA}_i(N, E', c, \mathcal{P}) + CEA_i(P_k, CEA_k(R, E - E', (c')^{\mathcal{P}}), (c'_j)_{j\in P_k}) \\ &= \overline{CEA}_i(N, E', c, \mathcal{P}) + \overline{CEA}_i(N, E - E', c', \mathcal{P}), \end{aligned}$$

where in the second equality we again use that CEA satisfies composition. Hence, we conclude that $\overline{CEA}$ satisfies composition. The proof of invariance under claims truncation within the unions follows similar lines.

To show the reverse, let $\varphi : BU^N \to \mathbb{R}^N$ be a rule satisfying ET, QPP, COMP and ICT. Let $(N, E, c, \mathcal{P}) \in BU^N$ and consider the quotient problem $(R, E, c^{\mathcal{P}}, \mathcal{P}^R)$. Without loss of generality, assume that $0 \leq c_1^{\mathcal{P}} \leq \ldots \leq c_r^{\mathcal{P}}$. In Proposition 1 of Dagan (1996) it is established that the constrained equal award rule is the only rule for bankruptcy situations that satisfies the bankruptcy equivalents of ET, COMP

and ICT. Since the quotient problem with $\mathcal{P}^R$ is basically a bankruptcy situation, it follows that $\varphi_k(R, E, c^{\mathcal{P}}, \mathcal{P}^R) = E_k^{CEA}$ for all $k \in R$.
Now, we consider the first union $P_1 \in \mathcal{P}$. Suppose without loss of generality that $P_1 = \{1, \ldots, n_1\}$ and that $c_{11} \leq \ldots \leq c_{1n_1}$.

Step 1. If $0 \leq E \leq rc_{11}$, then $E_1^{CEA} \leq c_{11}$ and because of ICT, QPP and ET, $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_1$.

If $rc_{11} < E \leq rc_{11} + rc_{11}(1 - \frac{1}{n_1})$, then equality is established using COMP. Next, COMP can be used for $rc_{11} + rc_{11}(1 - \frac{1}{n_1}) < E \leq rc_{11} + rc_{11}(1 - \frac{1}{n_1}) + rc_{11}(1 - \frac{1}{n_1})^2$. Repeating the same construction infinitely many times, $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_1$ if $0 \leq E \leq rn_1c_{11}$.

Step 2. If $rn_1c_{11} < E \leq rn_1c_{11} + r(c_{12} - c_{11})$, by COMP we have $\varphi(N, E, c, \mathcal{P}) = x + \varphi(N, E - rn_1c_{11}, c - x, \mathcal{P})$, where as a result of Step 1, $x_i = \varphi_i(N, rn_1c_{11}, c, \mathcal{P}) = \overline{CEA}_i(N, rn_1c_{11}, c, \mathcal{P}) = c_{11}$ for all $i \in P_1$. Furthermore, $E - rn_1c_{11} \leq r(c_{12} - c_{11})$. So because of QPP (keeping in mind that between the unions we have the CEA solution), ICT and ET we have $\varphi_i(N, E - rn_1c_{11}, c - x, \mathcal{P}) = \overline{CEA}_i(N, E - rn_1c_{11},\ c - x, \mathcal{P})$ for all $i \in P_1$ and hence, $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_1$.

Using a similar repetitive procedure as in Step 1, we obtain $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_1$ if $0 \leq E \leq rn_1c_{11} + r(n_1 - 1)(c_{12} - c_{11})$.

Repeating the same arguments, we conclude $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_1$ if $0 \leq E \leq rn_1c_{11} + r(n_1 - 1)(c_{12} - c_{11}) + \ldots + r(c_{1n_1} - c_{1,n_1-1}) = r(c_{11} + \ldots + c_{1n_1}) = rc_1^{\mathcal{P}}$. It then follows from $\varphi_1(R, E, c^{\mathcal{P}}, \mathcal{P}^R) = E_1^{CEA}$ and QPP that $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $0 \leq E \leq c(N)$.
Now, we consider the second union $P_2 \in \mathcal{P}$. We distinguish between two cases. If $E \leq rc_1^{\mathcal{P}}$, then we can use the same arguments as in the first union to obtain $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_2$.
So, assume that $E > rc_1^{\mathcal{P}}$. Because φ satisfies COMP, we have

$$\varphi(N, E, c, \mathcal{P}) = \varphi(N, rc_1^{\mathcal{P}}, c, \mathcal{P}) + \varphi(N, E - rc_1^{\mathcal{P}}, c - x, \mathcal{P}),$$

where $x = \varphi(N, rc_1^{\mathcal{P}}, c, \mathcal{P})$. By the previous case, $\varphi_i(N, rc_1^{\mathcal{P}}, c, \mathcal{P}) = \overline{CEA}_i(N, rc_1^{\mathcal{P}}, c, \mathcal{P})$ for all $i \in P_2$. With the second term, $\varphi(N, E - rc_1^{\mathcal{P}}, c - x, \mathcal{P})$, we proceed as with the first union with estate $E - rc_1^{\mathcal{P}}$ and claims $c - x$ and we obtain $\varphi_i(N, E - rc_1^{\mathcal{P}}, c - x, \mathcal{P}) = \overline{CEA}_i(N, E - rc_1^{\mathcal{P}}, c - x, \mathcal{P})$ for all $i \in P_2$. Note that in

the problem $(N, E - rc_1^{\mathcal{P}}, c - x, \mathcal{P})$ all the members of P_1 obtain zero. Because $\overline{CEA}$ satisfies COMP, we have $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_2$.
By repeating the same argument for all the unions, the statement follows. $\square$

Our second characterisation is based on Herrero and Villar (2002). In order to give this result, we first present some lemmas.

Lemma 10.4.2 *If $\varphi : BU^N \to \mathbb{R}^N$ is a rule that satisfies path independence and sustainability of players within the unions then for every $(N, E, c, \mathcal{P}) \in BU^N$ we have that $\varphi_k(R, E, c^{\mathcal{P}}, \mathcal{P}^R) = E_k^{CEA}$ for all $k \in R$.*

Proof: Let $\varphi : BU^N \to \mathbb{R}^N$ be a rule satisfying PI and SUS and let $(N, E, c, \mathcal{P}) \in BU^N$. Consider the associated quotient problem $(R, E, c^{\mathcal{P}}, \mathcal{P}^R)$. Theorem 1 in Herrero and Villar (2002) states that the constrained equal award rule is the only rule for bankruptcy situations that satisfies the bankruptcy equivalents of path independence and sustainability. From this, the statement readily follows. $\square$

Lemma 1 in Herrero and Villar (2002) states that if a bankruptcy rule satisfies path independence and sustainability, then it satisfies equal treatment of equals. In a similar way we can establish the next result for a rule for bankruptcy situations with a priori unions.

Lemma 10.4.3 *If a rule for bankruptcy problems with a priori unions satisfies the quotient problem property, path independence and sustainability within the unions, then it satisfies equal treatment within the unions.*

Now we can give our second axiomatic characterisation of the $\overline{CEA}$ rule.

Theorem 10.4.4 *The $\overline{CEA}$ rule is the unique rule for bankruptcy problems with a priori unions that satisfies path independence, sustainability of players within the unions and the quotient problem property.*

Proof: It is readily seen that $\overline{CEA}$ satisfies sustainability of players within the unions and the quotient problem property. The proof for path independence follows similar lines to the proof for composition and therefore we omit it.

Next, let $\varphi : BU^N \to \mathbb{R}^N$ be a rule for BU^N satisfying QPP, PI and SUS and let $(N, E, c, \mathcal{P}) \in BU^N$. Let $P_k \in \mathcal{P}$. We have to show that $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_k$. As a result of Lemma 10.4.2 and QPP, we have $\sum_{i \in P_k} = \varphi_i(N, E, c, \mathcal{P}) = E_k^{CEA}$. We use the following notation: $n_1^k = \max_{i \in P_k} c_i$, $N_1^k = \{i \in P_k \,|\, c_i = n_1^k\}$, $n_2^k = \max_{i \in P_k \setminus N_1^k} c_i$, $N_2^k = \{i \in P_k \,|\, c_i = n_2^k\}$.

Step 1. Suppose all claims in P_k are sustainable. Then $c_k^{\mathcal{P}} = E_k^{CEA}$ and because φ satisfies SUS, $\varphi_i(N, E, c, \mathcal{P}) = c_i = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_k$.

Step 2. Suppose only the claims of the players in $P_k \setminus N_1^k$ are sustainable in P_k. (If $P_k = N_1^k$, then we can immediately apply ET.) Then $\varphi_i(N, E, c, \mathcal{P}) = c_i$ for all $i \in P_k \setminus N_1^k$ and as a result of ET (using Lemma 10.4.3), the members of N_1^k all receive the same amount, which is at least n_2^k because this claim is sustainable. Using QPP, we obtain $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_k$.

Step 3. Next, suppose only the claims of the players in $P_k \setminus (N_1^k \cup N_2^k)$ are sustainable in P_k. Let $E' > E$ be such that $\varphi_k(R, E', c^{\mathcal{P}}, \mathcal{P}^R)$ is the minimum quantity that sustains the claims of $P_k \setminus N_1^k$ within union P_k, which is possible because of Lemma 10.4.2 and the basic properties of CEA. Let $c' = \varphi(N, E', c, \mathcal{P})$. By Step 1, $c'_i = c_i$ for all $i \in P_k \setminus N_1^k$ and because we chose E' minimal, $c'_i = c'_j$ for all $i, j \in N_1^k \cup N_2^k$. Because φ and $\overline{CEA}$ satisfy PI, we have $\varphi_i(N, E, c, \mathcal{P}) = \varphi_i(N, E, c', \mathcal{P})$ and $\overline{CEA}_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c', \mathcal{P})$ for all $i \in N$. By Step 1, $\varphi_i(N, E, c', \mathcal{P}) = \overline{CEA}_i(N, E, c', \mathcal{P})$ for all $i \in P_k$ and hence, $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_k$.

Repeating this procedure, we obtain $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_k$. □

10.5 Consistent two-step rules

In this section we define the second two-step extension of bankruptcy rules to bankruptcy situations with a priori unions. As in section 3, we use the CEA rule to illustrate this new extension and hence we obtain a second extension of the CEA rule for bankruptcy situations to bankruptcy situations with a priori unions, which we call RA^{CEA}. We also introduce a property of consistency which we subsequently use to characterise this extension.

Let $f : BR^N \to \mathbb{R}^N$ be a game-theoretic bankruptcy rule and let $(N, E, c, \mathcal{P}) \in BU^N$. We define the *$f$-random arrival rule*, $RA^f : BU^N \to \mathbb{R}^N$, in the following way:

$$RA_i^f(N, E, c, \mathcal{P}) = \frac{1}{r!}\left[\sum_{\tau \in \Pi(R)} f_i(P_k, E_\tau, (c_j)_{j \in P_k})\right]$$

for all $i \in P_k$, where $E_\tau = \max\{\min\{c_k^{\mathcal{P}}, E - \sum_{\ell \in R, \tau^{-1}(\ell) < \tau^{-1}(k)} c_\ell^{\mathcal{P}}\}, 0\}$.

The interpretation of this rule is similar to that of other solutions inspired by ideas of random arrival (cf. O'Neill (1982)). Here, we assume that the claims of the different unions are satisfied following a fixed order. If at the moment to allocate money to a particular union, the remaining estate is not enough to satisfy its total claim, we use the rule f to distribute the money available within this union. The f-random arrival rule allocates to a player the average of the amounts he obtains according to the previous procedure over all the possible orders on the unions.

Note that if $\mathcal{P} = \mathcal{P}^n$, then we have $RA^f(N, E, c, \mathcal{P}^n) = RTB(N, E, c)$, that is, in this boundary case, RA^f coincides with the run-to-the-bank rule for bankruptcy problems for every bankruptcy rule f. If $\mathcal{P} = \mathcal{P}^N$, then because f is game-theoretic the f-random arrival rule coincides with the rule f.

In the next example, we illustrate the CEA-random arrival rule.

Example 10.5.1 We compute RA^{CEA} in the bankruptcy situation with a priori unions of example 10.2.1. If the claims of the union P_1 are satisfied first, then the players obtain $(100, 100, 200)$, whereas if the claims of the union P_2 are satisfied first the players obtain $(0, 100, 300)$. Computing the average of the previous amounts, we obtain $RA^{CEA}(N, E, c, \mathcal{P}) = (50, 100, 250)$. Note that $RA^{CEA}(N, E, c, \mathcal{P})$ differs from both $\overline{CEA}(N, E, c, \mathcal{P})$ and $CEA(N, v_{E,c}, \mathcal{P})$. $\triangleleft$

Now, we define the property of consistency for bankruptcy with a priori unions rules which resembles the property of issue-consistency described in section 9.4. A rule $\varphi : BU^N \to \mathbb{R}^N$ is *consistent* if for every $(N, E, c, \mathcal{P}) \in BU^N$, for each union $P_k \in \mathcal{P}$ and for each player $i \in P_k$ we have

$$\varphi_i(N, E, c, \mathcal{P}) =$$
$$\tfrac{1}{r}\left[\varphi_i(P_k, E', (c_j)_{j \in P_k}, \mathcal{P}^{P_k}) + \textstyle\sum_{\ell \in R, \ell \neq k} \varphi_i(N \backslash P_\ell, E_{-\ell}, c_{-\ell}, \mathcal{P}_{-\ell})\right],$$

where $E' = \min\{E, c_k^{\mathcal{P}}\}$, $E_{-\ell} = \max\{E - c_\ell^{\mathcal{P}}, 0\}$, $c_{-\ell} = (c_j)_{j \in N \backslash P_\ell}$ and $\mathcal{P}_{-\ell}$ is the partition of the set $N \backslash P_\ell$ induced by $\mathcal{P}$.

So, a rule is consistent if in a bankruptcy problem with a priori unions it allocates to a player the average of what he gets when the rule is applied to the problem restricted to his own union and the solutions of the $r-1$ bankruptcy situations in which the estate is the amount that remains when each of the other unions gets its maximum. Note that if $\mathcal{P} = \mathcal{P}^n$, this definition of consistency corresponds to O'Neill consistency as defined in section 8.6.

Let $f : BR^N \to \mathbb{R}^N$ be a game-theoretic bankruptcy rule. We say that a consistent rule $\varphi : BU^N \to \mathbb{R}^N$ is *f-consistent* if for every bankruptcy problem $(N, E, c) \in BR^N$ we have that $\varphi(N, E, c, \mathcal{P}^N) = f(N, E, c)$. That is, a rule is f-consistent if it is consistent and it coincides with f when the a priori unions structure $\mathcal{P}$ is the boundary system $\mathcal{P}^N$.

The next theorem establishes, for a fixed bankruptcy rule f, the existence and uniqueness of an f-consistent rule. This result extends the O'Neill result of existence and uniqueness of a bankruptcy consistent rule (the run-to-the-bank rule).

Theorem 10.5.1 *For each game-theoretic bankruptcy rule $f : BR^N \to \mathbb{R}^N$, the f-random arrival rule RA^f is the unique f-consistent rule for bankruptcy situations with a priori unions.*

Proof: Let $f : BR^N \to \mathbb{R}^N$ be a game-theoretic bankruptcy rule. First we show that the f-random arrival rule, RA^f, is f-consistent. We know that for every bankruptcy problem $(N, E, c) \in BR^N$, $RA^f(N, E, c, \mathcal{P}^N) = f(N, E, c)$. So, it remains to be shown that RA^f is consistent. Let $(N, E, c, \mathcal{P}) \in BU^N$ and let $i \in P_k, k \in R$. Define E_σ, E' and $E_{-\ell}$ as before and $E_{-\ell,\tau} = \max\{\min\{c_k^{\mathcal{P}}, E_{-\ell} - \sum_{t\in R\setminus\{\ell\}:\tau^{-1}(t)<\tau^{-1}(k)} c_t^{\mathcal{P}}\}, 0\}$ for all $\tau \in \Pi(R)$, $\ell \in R$. Then,

$$
\begin{aligned}
RA_i^f(N, E, c, \mathcal{P}) &= \frac{1}{r!} \sum_{\tau\in\Pi(R)} f_i(P_k, E_\tau, (c_j)_{j\in P_k}) \\
&= \frac{1}{r!}\Big[(r-1)! f_i(P_k, E', (c_j)_{j\in P_k}) \\
&\quad + \sum_{\ell\in R,\ell\neq k} \sum_{\tau\in\Pi(R\setminus\{\ell\})} f_i(P_k, E_{-\ell,\tau}, (c_j)_{j\in P_k})\Big] \\
&= \frac{1}{r}\Big[f_i(P_k, E', (c_j)_{j\in P_k}) \\
&\quad + \sum_{\ell\in R,\ell\neq k} \frac{1}{(r-1)!} \sum_{\tau\in\Pi(R\setminus\{\ell\})} f_i(P_k, E_{-\ell,\tau}, (c_j)_{j\in P_k})\Big]
\end{aligned}
$$

$$= \frac{1}{r}\Big[RA_i^f(P_k, E', (c_j)_{j\in P_k}, \mathcal{P}^{P_k}) \\ + \sum_{\ell\in R,\, \ell\neq k} RA_i^f(N\backslash P_\ell, \max\{E - c_\ell^{\mathcal{P}}, 0\}, c_{-\ell}, \mathcal{P}_{-\ell})\Big].$$

Hence, RA^f is consistent and therefore f-consistent.

Uniqueness of the f-consistent rule follows from a similar recursive argument as in the proof of Theorem 8.6.1. □

O'Neill (1982) shows that the random arrival (run-to-the-bank) solution of a bankruptcy situation coincides with the Shapley value of the corresponding bankruptcy game. The next theorem extends this result by O'Neill to our context, in the sense that the RA-random arrival rule coincides with the *Owen value* (cf. Owen (1977)) of the corresponding bankruptcy game with a priori unions. We omit the proof, which follows a similar line to the proof of the preceding theorem.

Theorem 10.5.2 *Let* $(N, E, c, \mathcal{P}) \in BU^N$. *Then*

$$RA^{RA}(N, E, c, \mathcal{P}) = Ow(N, v_{E,c}, \mathcal{P}).$$

Now, from the previous two theorems, we immediately obtain the following result.

Theorem 10.5.3 *The only rule for bankruptcy situations with a priori unions satisfying random arrival-consistency is the Owen value of the associated bankruptcy games with a priori unions.*

In Winter (1992) and Hamiache (1999), the Owen value is axiomatically characterised on the class of cooperative games with a priori unions by using two different properties of consistency. Note that in the current chapter, we characterise the Owen value on the class of bankruptcy situations with a priori unions, using a different consistency property which extends the O'Neill consistency property for bankruptcy situations.

Chapter 11

A characterisation of the τ value

11.1 Introduction

Most game-theoretic solution concepts that have been proposed in the literature are defined on the basis of or characterised by properties. These properties are usually formulated in terms of individual payoffs and reflect notions like monotonicity and rationality. For some values, there exist additional characterisations in terms of geometry. The best-known example is the Shapley value, which is the barycentre of the extreme points of the Weber set (taking multiplicities into account).

For some classes of games, there exist nice geometric expressions for the compromise value. In particular, the compromise value is the barycentre of the core cover in big boss games (cf. Muto et al. (1988)) and 1-convex games (cf. Driessen (1988)).

In this chapter, which is based on González-Diaz et al. (2003), we extend the $APROP$ rule for bankruptcy situations to the whole class of compromise admissible games. This extended rule, which we call τ^*, turns out to be the barycentre of the *edges* of the core cover (taking multiplicities into account). Since this rule coincides with the compromise value if, after normalising such that each player's minimal right equals zero, each player's utopia payoff is at most the value of the grand coalition, our main result immediately provides a characterisation of the compromise value on this class of games.

This chapter is organised as follows. In section 2, we extend the $APROP$ rule to the class of compromise admissible games and define the barycentre ζ of the edges of the core cover. In section 3, we state our main result and give an overview of the proof, which consists of six main steps. Finally, in section 4, we prove our result.

11.2 The τ^* value

The literature offers many different bankruptcy rules (see section 7.2) and hence, indirectly, rules for bankruptcy games. One interesting question is how these can be extended in a natural way to the whole class of compromise admissible games. In this chapter, we consider the proportional rule and the adjusted proportional rule[1]. Recall that the proportional rule $PROP$ simply divides the estate proportional to the claims:

$$PROP_i(E,c) = \frac{c_i}{\sum_{j\in N} c_j} E$$

for all $(N,E,c) \in BR^N$ and $i \in N$. The adjusted proportional rule $APROP$ first gives each player $i \in N$ his minimal right $m_i(E,c) = \max\{E - \sum_{j\in N\setminus\{i\}} c_j, 0\}$ and the remainder is divided using the proportional rule, where each player's claim is truncated to the estate left:

$$APROP(E,c) = m(E,c) + PROP(E',c'),$$

where $E' = E - \sum_{i\in N} m_i(E,c)$ and for all $i \in N$, $c'_i = \min\{c_i - m_i(E,c), E'\}$.

The compromise value can be seen as an extension of the $PROP$ rule:

$$\tau(v) = m(v) + PROP(v(N) - \sum_{i\in N} m_i(v), M(v) - m(v)).$$

Note that it follows from the definition of compromise admissibility that the argument of $PROP$ is indeed a bankruptcy situation.

Similarly, we can extend the $APROP$ rule:

$$\tau^*(v) = m(v) + APROP(v(N) - \sum_{i\in N} m_i(v), M(v) - m(v)).$$

To simplify the expression for τ^*, we show that the minimum rights in the associated bankruptcy situation equal 0. Let $v \in CA^N$, $E = v(N) - \sum_{i\in N} m_i(v)$, $c = M(v) - m(v)$ and let $i \in N$. Then

$$\begin{aligned} E - \sum_{j\in N\setminus\{i\}} c_j &= v(N) - \sum_{i\in N} m_i(v) - \sum_{j\in N\setminus\{i\}} (M_j(v) - m_j(v)) \\ &= v(N) - m_i(v) - \sum_{j\in N\setminus\{i\}} M_j(v) \\ &\leq 0, \end{aligned}$$

[1]The extension of the Talmud rule is discussed in Quant et al. (2003), while in Quant et al. (2004) a more general framework is considered, including the run-to-the-bank rule, the constrained equal award rule and the constrained equal loss rule.

since $m_i(v) \geq v(N) - \sum_{j \in N \setminus \{i\}} M_j(v)$. Hence, $m_i(E,c) = \max\{E - \sum_{j \in N \setminus \{i\}} c_j, 0\} = 0$. As a result, we have

$$\tau^*(v) = m(v) + PROP(E', c') \tag{11.1}$$

with $E' = v(N) - \sum_{i \in N} m_i(v)$ and $c'_i = \min\{M_i(v) - m_i(v), E'\}$ for all $i \in N$.

It follows that for a game $v \in CA^N$ with $M_i(v) - m_i(v) \leq v(N) - \sum_{j \in N} m_j(v)$ for all $i \in N$, τ^* coincides with the compromise value τ.

The extended rule τ^* turns out to be a kind of barycentre of the core cover, which is the main result of this chapter. To define this barycentre rule ζ, we need to introduce some more concepts. For a permutation[2] $\sigma \in \Pi(N)$, $\sigma^{i,j}$ denotes the permutation obtained from σ by switching players i and j. Two permutations σ and $\sigma^{\sigma(p),\sigma(p+1)}$ are called *permutation neighbours*. The set of permutation neighbours of σ is denoted by $\Pi^\sigma(N)$.

The core cover is a polytope whose extreme points are called *larginal vectors* or *larginals*. The larginal $\ell^\sigma \in \mathbb{R}^N$ corresponding to order $\sigma \in \Pi(N)$ (cf. Quant et al. (2003)) is given by

$$\ell^\sigma_{\sigma(p)}(v) = \begin{cases} M_{\sigma(p)}(v) \text{ if } \sum_{k=1}^{p} M_{\sigma(k)}(v) + \sum_{k=p+1}^{n} m_{\sigma(k)}(v) \leq v(N), \\ m_{\sigma(p)}(v) \text{ if } \sum_{k=1}^{p-1} M_{\sigma(k)}(v) + \sum_{k=p}^{n} m_{\sigma(k)}(v) > v(N), \\ v(N) - \sum_{k=1}^{p-1} M_{\sigma(k)}(v) - \sum_{k=p+1}^{n} m_{\sigma(k)}(v) \text{ otherwise} \end{cases}$$

for all $p \in \{1, \ldots, n\}$.

Note that two permutations that are neighbours yield larginals which are adjacent extreme points of the core cover (possibly coinciding), which we subsequently also call permutation neighbours.

We define the ζ rule as a weighted average of the larginal vectors:[3]

$$\zeta(v) = \frac{\sum_{\sigma \in \Pi(N)} w^\sigma(v) \ell^\sigma(v)}{\sum_{\sigma \in \Pi(N)} w^\sigma(v)}, \tag{11.2}$$

where

$$w^\sigma(v) = \frac{1}{\sqrt{2}} \sum_{\tau \in \Pi^\sigma(N)} d(\ell^\sigma(v), \ell^\tau(v))$$

[2] In this chapter, we use the term *permutation* rather than *ordering* for an element of $\Pi(N)$

[3] In the degenerate case where $M = m$, the core cover consists of a single point, in which case we define ζ to be this point. Otherwise, there are at least two different larginals and the denominator is positive.

equals the sum of the Euclidean distances between $\ell^\sigma(v)$ and all its permutation neighbours, divided by the common factor $\sqrt{2}$ to simplify later expressions. The ζ value can be viewed as the barycentre of the *edges* of the core cover, taking multiplicities into account.

To simplify the proofs later on by getting rid of the minimum rights and capping the utopia payoffs, we first show that both τ^* and ζ satisfy the properties (SEQ) and (RTRUNC). Two games v and $\hat{v}$ are called strategically equivalent if there exists a real number $k > 0$ and a vector $a \in \mathbb{R}^N$ such that for all $S \subset N$,

$$\hat{v}(S) = kv(S) + a(S). \tag{11.3}$$

A function $f : CA^N \to \mathbb{R}^N$ is *relatively invariant with respect to strategic equivalence* (SEQ) if for all $v, \hat{v} \in CA^N$ such that for all $S \subset N$ (11.3) holds for some $k > 0, a \in \mathbb{R}^N$, we have

$$f(\hat{v}) = kf(v) + a.$$

It is well-known that the utopia vector M and the minimum right vector m both satisfy (SEQ).

Proposition 11.2.1 *The τ^* rule and the ζ rule satisfy (SEQ).*

Proof: The proof for τ^* is straightforward and therefore omitted.
It readily follows from (SEQ) of m and M that ℓ^σ also satisfies (SEQ) for all $\sigma \in \Pi(N)$. Let $v, \hat{v} \in CA^N$ be such that for some $k > 0, a \in \mathbb{R}^N$ (11.3) holds for all $S \subset N$ and let $\sigma \in \Pi(N)$. Then

$$\begin{aligned}
w^\sigma(\hat{v}) &= \frac{1}{\sqrt{2}} \sum_{\tau \in \Pi^\sigma(N)} d(\ell^\sigma(\hat{v}), \ell^\tau(\hat{v})) \\
&= \frac{1}{\sqrt{2}} \sum_{\tau \in \Pi^\sigma(N)} d(k\ell^\sigma(v) + a, k\ell^\tau(v) + a) \\
&= k\frac{1}{\sqrt{2}} \sum_{\tau \in \Pi^\sigma(N)} d(\ell^\sigma(v), \ell^\tau(v)) \\
&= kw^\sigma(v)
\end{aligned}$$

Hence,

$$\begin{aligned}\zeta(\hat{v}) &= \frac{\sum_{\sigma\in\Pi(N)} w^\sigma(\hat{v})\ell^\sigma(\hat{v})}{\sum_{\sigma\in\Pi(N)} w^\sigma(\hat{v})} \\ &= \frac{k\sum_{\sigma\in\Pi(N)} w^\sigma(v)[k\ell^\sigma(v)+a]}{k\sum_{\sigma\in\Pi(N)} w^\sigma(v)} \\ &= k\zeta(v)+a.\end{aligned}$$

And so, ζ satisfies (SEQ). □

A rule $f : CA^N \to \mathbb{R}^N$ satisfies the *restricted truncation property* (RTRUNC) if for all $v \in CA^N$ with $m(v) = 0$ it holds that for all $\hat{v} \in CA^N$ with $\hat{v}(N) = v(N)$, $m(\hat{v}) = 0$ and $M_i(\hat{v}) = \min\{M_i(v), v(N)\}$ we have $f(\hat{v}) = f(v)$. The idea behind (RTRUNC) is that if a player's utopia value (or, in bankruptcy terms, his claim) is higher than the value of the grand coalition (the estate), his payoff according to f should not be influenced by truncating this claim.

Proposition 11.2.2 *The τ^* rule and the ζ rule satisfy (RTRUNC).*

Proof: Let $v \in CA^N$ with $m(v) = 0$. Then (11.1) reduces to

$$\tau^*(v) = PROP(v(N), (\min\{M_i(v), v(N)\})_{i\in N}).$$

From this it immediately follows that τ^* satisfies (RTRUNC).
For the ζ rule, it suffices to note that truncating the utopia vector has no influence on the larginal vectors. □

11.3 Main result

In this section, we present the main result of this chapter: equality between τ^* and ζ on CA^N. After dealing with some simple cases, we present a six step outline of the proof, which we give in the next section.

Theorem 11.3.1 *Let $v \in CA^N$. Then*

$$\tau^*(v) = \zeta(v).$$

As a result of Proposition 11.2.1, it suffices to show equality for every game $v \in CA^N$ with $m(v) = 0$. Next, we can use Proposition 11.2.2 and conclude that we have to show that for all $v \in CA^N$ with $m(v) = 0$ and $M_i(v) \leq v(N)$[4] for all $i \in N$ we have[5]

$$PROP(v(N), M(v)) = \frac{\sum_{\sigma \in \Pi(N)} w^\sigma(v)\ell^\sigma(v)}{\sum_{\sigma \in \Pi(N)} w^\sigma(v)}.$$

In case there are only two players, equality between τ^* and ζ follows from $M_1(v) = M_2(v) = v(N)$.

If $M_i(v) = 0$ for a player $i \in N$, then we have $\tau_i^*(v) = \zeta_i(v) = 0$. Furthermore, for each $\sigma \in \Pi(N)$, the payoff to the players in $N\backslash\{i\}$ according to $\ell^\sigma(v)$ equals their payoff in the situation without player i[6] according to the larginal corresponding to the restricted permutation $\sigma_{N\backslash\{i\}} \in \Pi(N\backslash\{i\})$, defined by $\sigma^{-1}_{N\backslash\{i\}}(h) < \sigma^{-1}_{N\backslash\{i\}}(j) \Leftrightarrow \sigma^{-1}(h) < \sigma^{-1}(j)$ for all $h, j \in N\backslash\{i\}$. It is readily verified that also the total weight of each larginal (taking multiplicities into account) is the same in the game with and without player i. Hence, we can omit player i from the game and establish equality between τ^* and ζ for the remaining players.[7]

We establish equality between τ^* and ζ by combining the permutations in the numerator and denominator in (11.2) into so-called *chains*. In the denominator, these chains allow us to combine terms in such a way that the total weight can be expressed as a simple function of $M(v)$. In the numerator, we construct an iterative procedure to find an expression for the weighted larginals, in which the chains allow us to keep track of changes that occur from one iteration to the next.

The proof of Theorem 11.3.1 consists of six steps:

1. We first find an expression for the weight of each permutation. This is done by introducing the concept of *pivot* and classifying each permutation in terms of its pivot and its neighbours' pivots.

2. Using the concept of pivot, we introduce *chains*, which constitute a partition of the set of all permutations. The results of the previous step are then used to compute the total weight of each chain.

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[4] Note that the condition $M_i(v) \leq v(N)$ is necessary and sufficient to have $M_i(v) = \max_{\sigma \in \Pi(N)} \ell_i^\sigma(v)$. Only in this case, the utopia vector can be reconstructed from the core cover.

[5] The denominator is zero if and only if $M(v) = 0$ $(= m(v))$. In this degenerate case equality between τ^* and ζ is trivial and we therefore assume $M(v) \gneqq 0$.

[6] Ie, the situation with player set $N\backslash\{i\}$, utopia vector $M_{N\backslash\{i\}}(v)$ and the same amount $v(N)$ to be distributed.

[7] Geometrically, the core cover, which lies in the hyperplane $M_i(v) = 0$, is projected onto a space which is one dimension lower.

3. We define a family of auxiliary functions f^{ij} and g^{ij}, which are used to show that each player "belongs" to the same number of chains. As a result, we use our expression of the previous step to compute the total of all the weights, ie, the denominator in (11.2).

4. In the numerator, we partition the set of chains on the basis of the first player in each permutation. Within each part, we compute the total weighted payoff to all the players. For the first player, this total weighted payoff can easily be computed.

5. The expression for the payoffs to the other players is proved using an iterative argument, varying the utopia vector while keeping $v(N)$ constant. We start with a utopia vector for which our expression is trivial and lower this vector step by step until we reach $M(v)$. In each step of the iteration, (generically) only two chains change and using this, we show that the total weighted payoff to each player who is not first does not change as function of the utopia vector.

6. Combining the previous three steps, we derive an expression for ζ and show that this equals τ^*.

11.4 Proof of main result

Throughout this section, let $v \in CA^N$ be such that $|N| \geq 3$, $m(v) = 0$, $M(v) > 0$, and $v(N) \geq M_i(v)$ for all $i \in N$. For Theorem 11.3.1 is suffices to show that for this v we have

$$PROP(v(N), M(v)) = \frac{\sum_{\sigma \in \Pi(N)} w^\sigma(v) \ell^\sigma(v)}{\sum_{\sigma \in \Pi(N)} w^\sigma(v)}.$$

Since v is fixed for the remainder of this section, we suppress it as argument and write M rather than $M(v)$, etc. The weight $w^\sigma(v)$ is denoted by $w(\sigma)$.

Step 1: pivots

Let $\sigma \in \Pi(N)$. Player $\sigma(p)$ with $p \geq 2$ is called the *pivot* in ℓ^σ if $\ell^\sigma_{\sigma(p-1)} = M_{\sigma(p-1)}$, $\ell^\sigma_{\sigma(p)} > 0$ and $\ell^\sigma_{\sigma(p+1)} = 0$. The pivot of a larginal is the player who gets a lower amount according this larginal if the amount $v(N)$ is decreased slightly. In the boundary case where $M_{\sigma(1)} = v(N)$, $v(N)$ cannot be decreased without violating the condition $M_{\sigma(1)} \leq v(N)$. In this case, player $\sigma(2)$ is defined to be the pivot, being the player who gets a higher amount if $v(N)$ is increased slightly. Note that $m = 0$ implies that $\sum_{j \in N\setminus\{i\}} M_j \geq v(N)$ for all $i \in N$ and hence, player $\sigma(n)$ can never be the pivot.

In the following example, we introduce a game which we use throughout this section to illustrate the various concepts.

Example 11.4.1 Consider the game (N, v) with $N = \{1, \ldots, 5\}$, $v(N) = 10$ and $M = (5, 7, 1, 3, 4)$. For this game, we have $\tau^* = \zeta = \frac{1}{2}M$. Take σ_1 to be the identity permutation, ie, $\sigma_1(i) = i$ for all $i \in N$. Then

$$\ell^{\sigma_1} = (5, 5, 0, 0, 0)$$

and player 2 is the pivot. $\triangleleft$

For a permutation $\sigma \in \Pi(N)$, we define p_σ to be the position at which the pivot[8] is located. We define $\sigma^L = \sigma^{\sigma(p_\sigma - 1), \sigma(p_\sigma)}$ to be the *left neighbour* of σ and $\sigma^R = \sigma^{\sigma(p_\sigma), \sigma(p_\sigma + 1)}$ to be the *right neighbour* of σ. It follows from the definition of pivot that the left and right neighbours of ℓ^σ are the only two permutation neighbours that can give rise to a larginal different from ℓ^σ.

Recall that the weight of ℓ^σ, $w(\sigma)$, equals the sum of the (Euclidean) distances between ℓ^σ and all its permutation neighbours. In line with the previous paragraph, we only have to take the left and right neighbours into account. So,

$$w(\sigma) = \frac{1}{\sqrt{2}} \left[d(\ell^\sigma, \ell^{\sigma^L}) + d(\ell^\sigma, \ell^{\sigma^R}) \right].$$

We classify the larginals into four categories, depending on the pivot in the left and right neighbours. Let $\sigma = (\ldots, h, i, j, \ldots)$ be a permutation with pivot i. Then the four types are given in the following table:

Type	Pivot in σ^L	Pivot in σ	Pivot in σ^R
PPP	i	i	i
$-PP$	h	i	i
$PP-$	i	i	j
$-P-$	h	i	j

We can now determine the weight of each larginal, depending on its type. Take $\sigma \in \Pi(N)$ to be the identity permutation and assume that ℓ^σ is of type $PP-$ and has pivot i. Then

[8] As with *neighbour*, we use the term *pivot* as property of a permutation as well as the corresponding larginal.

$$\begin{aligned}
\ell^{\sigma} &= (M_1, \ldots, M_{i-2}, M_{i-1}, v(N) - \sum_{j=1}^{i-1} M_j, 0, \ldots, 0), \\
\ell^{\sigma^L} &= (M_1, \ldots, M_{i-2}, 0, v(N) - \sum_{j=1}^{i-2} M_j, 0, \ldots, 0), \\
\ell^{\sigma^R} &= (M_1, \ldots, M_{i-2}, M_{i-1}, 0, v(N) - \sum_{j=1}^{i-1} M_j, 0, \ldots, 0).
\end{aligned}$$

So,

$$\begin{aligned}
d(\ell^{\sigma}, \ell^{\sigma^L}) &= \sqrt{2M_{i-1}^2} = \sqrt{2}M_{i-1}, \\
d(\ell^{\sigma}, \ell^{\sigma^R}) &= \sqrt{2(v(N) - \sum_{j=1}^{i-1} M_j)^2} = \sqrt{2}(v(N) - \sum_{j=1}^{i-1} M_j), \\
w(\sigma) &= (v(N) - \sum_{j=1}^{i-2} M_j).
\end{aligned}$$

Doing these calculations for all types and arbitrary $\sigma \in \Pi(N)$, we obtain the following weights:

Type	$w(\sigma)$
PPP	$M_{\sigma(p_\sigma-1)} + M_{\sigma(p_\sigma+1)}$
$-PP$	$\sum_{k=1}^{p_\sigma+1} M_{\sigma(k)} - v(N)$
$PP-$	$v(N) - \sum_{k=1}^{p_\sigma-2} M_{\sigma(k)}$
$-P-$	$M_{\sigma(p_\sigma)}$

Example 11.4.2 With σ_1 the identity permutation, we have (the player with ˆ is the pivot):

$$\begin{array}{ll}
\sigma_1 = (1, \hat{2}, 3, 4, 5) & \ell^{\sigma_1} = (5, 5, 0, 0, 0) \\
\sigma_1^L = (2, \hat{1}, 3, 4, 5) & \ell^{\sigma_1^L} = (3, 7, 0, 0, 0) \\
\sigma_1^R = (1, 3, \hat{2}, 4, 5) & \ell^{\sigma_1^R} = (5, 4, 1, 0, 0)
\end{array}$$

So, ℓ^{σ_1} is of type $-PP$. The weight of σ_1 equals

$$\begin{aligned}
w(\sigma_1) &= \frac{1}{\sqrt{2}}\left[d(\sigma_1, \sigma_1^L) + d(\sigma_1, \sigma_1^R)\right] \\
&= 2 + 1 \\
&= 3.
\end{aligned}$$

Indeed, we have $w(\sigma_1) = \sum_{k=1}^{p_{\sigma_1}+1} M_{\sigma_1(k)} - v(N) = M_1 + M_2 + M_3 - v(N) = 5 + 7 + 1 - 10 = 3$, as the table shows. ◁

Step 2: chains

A *chain* of length q and with pivot i is a set of q permutations $\Gamma = \{\sigma_1, \ldots, \sigma_q\}$ such that

- $(\sigma_m)^R = \sigma_{m+1}$ for all $m \in \{1, \ldots, q-1\}$,
- i is the pivot in σ_m for all $m \in \{1, \ldots, q\}$,
- i is not the pivot in σ_1^L and σ_q^R.

If $q = 1$, then it follows from the definitions of the four types that σ_1 is of type $-P-$. If $q > 1$, then σ_1 is of type $-PP$, σ_m is of type PPP for all $m \in \{2, \ldots, q-1\}$ and σ_q is of type $PP-$. Observe that the set of all chains, which we denote by $\mathcal{C}$, constitutes a partition of the set of permutations $\Pi(N)$.

Denoting by σ^- the permutation on the $n-1$ players obtained from σ by removing the pivot, we characterise the chains in the following lemma.

Lemma 11.4.1 *Two permutations $\sigma_1, \sigma_2 \in \Pi(N)$ are in the same chain if and only if $\sigma_1^- = \sigma_2^-$.*

Given the weights of the larginal vectors, depending on their type, we can easily compute the weight of a chain Γ, which is simply defined as the total weight of its elements, ie, $w(\Gamma) = \sum_{\sigma \in \Gamma} w(\sigma)$.

Lemma 11.4.2 *Let $\Gamma = \{\sigma_1, \ldots, \sigma_q\} \in \mathcal{C}$. Then*

$$w(\Gamma) = \sum_{k=p_{\sigma_1}}^{p_{\sigma_1}+q-1} M_{\sigma_1(k)}.$$

Proof: Denoting $p = p_{\sigma_1}$, we have (for $q \geq 5$; for smaller chains the proof is similar):

$$\begin{array}{rcrcrcrcrcc}
w(\sigma_1) & = & \sum_{k=1}^{p-1} M_{\sigma_1(k)} & - & v(N) & + & M_{\sigma_1(p)} & + & M_{\sigma_1(p+1)} & & \\
w(\sigma_2) & = & & & & + & M_{\sigma_1(p+1)} & + & M_{\sigma_1(p+2)} & & \\
w(\sigma_3) & = & & & & + & M_{\sigma_1(p+2)} & + & M_{\sigma_1(p+3)} & & \\
\vdots & = & & & & & \vdots & & \vdots & & \\
w(\sigma_{q-1}) & = & & & & + & M_{\sigma_1(p+q-2)} & + & M_{\sigma_1(p+q-1)} & & \\
w(\sigma_q) & = & -\sum_{k=1}^{p-1} M_{\sigma_1(k)} & + & v(N) & - & \sum_{k=p+1}^{p+q-2} M_{\sigma_1(k)} & & & & + \\
\hline
w(\Gamma) & = & & & & & M_{\sigma_1(p)} & + & \sum_{k=p+1}^{p+q-1} M_{\sigma_1(k)} & &
\end{array}$$

□

We say that player $i \in N$ *belongs* to chain $\Gamma = \{\sigma_1, \ldots, \sigma_q\}$ if $i \in \{\sigma_1(p_{\sigma_1}), \ldots, \sigma_1(p_{\sigma_1}+q-1)\}$, ie, if his position is not constant throughout the chain. Alternatively, a player is said to belong to a chain if his utopia payoff contributes to its weight. We define $C(i)$ to be the set of chains to which i belongs. By $P(i) \subset C(i)$ we denote the set of chains in which i is the pivot and by $\bar{P}(i) = C(i) \backslash P(i)$ its complement. For each $\Lambda \in \bar{P}(i)$, we denote the permutation in Λ in which i is immediately before the pivot by λ_{bi} and the permutation in which i is immediately after the pivot by λ_{ai}.

Example 11.4.3 Since player 2 is not the pivot in σ_1^L, σ_1 is the first permutation of a chain. This chain Γ consists of σ_1, $\sigma_2 = \sigma_1^R$ and $\sigma_3 = \sigma_2^R$, all having player 2 as pivot. In line with Lemma 11.4.1, we have $\sigma_1^- = \sigma_2^- = \sigma_3^- = (1, 3, 4, 5)$. Players 2, 3 and 4 belong to Γ and $w(\Gamma) = M_2 + M_3 + M_4 = 11$. ◁

Step 3: denominator

In this step, we derive an expression for the denominator in (11.2). We do this by showing that each player belongs to the same number of chains, ie,

$$|C(i)| = |C(j)| \tag{11.4}$$

for all $i, j \in N$. If $M_i = M_j$, then this is trivial, so throughout this step, let $i, j \in N$ be such that $M_i > M_j$. We prove only one part of (11.4):

$$|P(j)| + |\bar{P}(j)| \leq |P(i)| + |\bar{P}(i)|. \tag{11.5}$$

The proof of the reverse inequality goes along similar lines, as will be indicated later on.

An immediate consequence of Lemma 11.4.4 below is that $|P(i)| \geq |P(j)|$ and $|\bar{P}(j)| \geq |\bar{P}(i)|$. In Proposition 11.4.5 below we partner all the chains in $P(j)$ to some of the chains in $P(i)$ and we partner all the chains $\bar{P}(i)$ to some of the chains in $\bar{P}(j)$. We then show that for every chain in $\bar{P}(j)$ which has no partner in $\bar{P}(i)$, we can find a chain in $P(i)$ which has no partner in $P(j)$. From this, (11.5) follows.

To partner the various chains, we define two auxiliary functions. First, we define f^{ij}:

$$\begin{array}{rcl} P(j) & \stackrel{f^{ij}}{\rightarrow} & P(i) \\ \Delta & \mapsto & f^{ij}(\Delta) = \Lambda \end{array}$$

where $\Delta = \{\delta_1, \ldots, \delta_q\}$ and Λ is the chain containing $\delta_1^{i,j}$. Note that the function f^{ij} is well-defined: since $M_i > M_j$, player i is indeed the pivot in $\delta_1^{i,j}$ and hence, in Λ.

Similarly, we define the function g^{ij}:

$$\begin{array}{rcl} \bar{P}(i) & \stackrel{g^{ij}}{\rightarrow} & \bar{P}(j) \\ \Lambda & \mapsto & g^{ij}(\Lambda) = \Delta \end{array}$$

where for all $\Lambda \in \bar{P}(i)$, Δ is the chain containing $\lambda_{bi}^{i,j}$.[9]

In the following lemma, we show that g^{ij} is well-defined, ie, that the chain Δ thus constructed is indeed an element of the range of g^{ij}, $\bar{P}(j)$.

Lemma 11.4.3 *The function g^{ij} is well-defined.*

Proof: Denote the pivot player in λ_{bi} (and hence, λ_{ai}) by h. Observe that as a result of $M_i > M_j$, player h cannot coincide with j. Distinguish between the following two cases:

- i is before j in λ_{bi}:

$$\begin{array}{ll} \lambda_{ai} = (\ldots, \hat{h}, i, \ldots, j, \ldots) & \lambda_{ai}^{i,j} = (\ldots, \hat{h}, j, \ldots, i, \ldots) \\ \lambda_{bi} = (\ldots, i, \hat{h}, \ldots, j, \ldots) & \lambda_{bi}^{i,j} = (\ldots, j, \hat{h}, \ldots, i, \ldots) \end{array}$$

  Since h is the pivot in λ_{ai}, it immediately follows that h is also the pivot in $\lambda_{ai}^{i,j}$. Player j cannot be the pivot in $\lambda_{bi}^{i,j}$, because i is before the pivot in λ_{bi} and $M_i > M_j$. Combining this with the fact that h is the pivot in $\lambda_{ai}^{i,j}$, h is also the pivot in $\lambda_{bi}^{i,j}$. But then $\lambda_{ai}^{i,j}$ belongs to the same chain Δ as $\lambda_{bi}^{i,j}$. From this, $\Delta \in C(j)$, and because j is not the pivot in Δ, $\Delta \in \bar{P}(j)$.

- j is before i in λ_{bi}:

[9] By $\lambda_{bi}^{i,j}$ we mean $(\lambda_{bi})^{i,j}$, ie, the permutation which is obtained by switching i and j in the permutation in Λ where i is immediately before the pivot.

$$\begin{array}{ll} \lambda_{ai} = (\ldots, j, \ldots, \hat{h}, i, \ldots) & \lambda_{ai}^{i,j} = (\ldots, i, \ldots, \hat{h}, j, \ldots) \\ \lambda_{bi} = (\ldots, j, \ldots, i, \hat{h}, \ldots) & \lambda_{bi}^{i,j} = (\ldots, i, \ldots, j, \hat{h}, \ldots) \end{array}$$

Since h is the pivot in λ_{bi}, we immediately have that h is the pivot in $\lambda_{bi}^{i,j}$. Because of this, the pivot in $\lambda_{ai}^{i,j}$ cannot be before h. It can also not be after h, because h is the pivot in λ_{ai} and $M_i > M_j$. By the same argument as in the first case, $\Delta \in \bar{P}(j)$.

From these two cases, we conclude that g^{ij} is well-defined. □

For our partnering argument to hold, we need that the functions f^{ij} and g^{ij} are injective. This is shown in the following lemma.

Lemma 11.4.4 *The functions f^{ij} and g^{ij} are injective.*

Proof: To see that f^{ij} is injective, let $\Delta, \tilde{\Delta} \in P(j)$ be such that $f^{ij}(\Delta) = f^{ij}(\tilde{\Delta})$. By construction, i is the pivot in both $f^{ij}(\Delta)$ and $f^{ij}(\tilde{\Delta})$, so i is the pivot in both $\delta_1^{i,j}$ and $\tilde{\delta}_1^{i,j}$. Since by assumption these permutations are in the same chain, by Lemma 11.4.1 we have $(\delta_1^{i,j})^- = (\tilde{\delta}_1^{i,j})^-$. But since j is the pivot in both δ_1 and $\tilde{\delta}_1$, it follows that $\delta_1^- = \tilde{\delta}_1^-$. So, δ_1 and $\tilde{\delta}_1$ are in the same chain and $\Delta = \tilde{\Delta}$.
For injectivity of g^{ij}, let $\Lambda, \tilde{\Lambda} \in \bar{P}(i)$ be such that $g^{ij}(\Lambda) = g^{ij}(\tilde{\Lambda})$. Then $\lambda_{bi}^{i,j}$ and $\tilde{\lambda}_{bi}^{i,j}$ are in the same chain. By the proof of Lemma 11.4.3, j is just before the pivot in both permutations and hence, $\lambda_{bi}^{i,j} = \tilde{\lambda}_{bi}^{i,j}$. From this, we conclude $\lambda_{bi} = \tilde{\lambda}_{bi}$ and $\Lambda = \tilde{\Lambda}$. □

From Lemma 11.4.4, we conclude

$$|P(j)| \leq |P(i)|$$

and

$$|\bar{P}(i)| \leq |\bar{P}(j)|.$$

With these inequalities, we can now apply our partnering argument to prove that each player belongs to the same number of chains.

Proposition 11.4.5 *Let $i, j \in N$. Then $|C(i)| = |C(j)|$.*

Proof: If $M_i = M_j$, then the statement is trivial. Hence, assume without loss of generality that $M_i > M_j$.
We only show (11.5). Let $\Delta \in \bar{P}(j)$ be such that there exists no $\Lambda \in \bar{P}(i)$ with $g^{ij}(\Lambda) = \Delta$. Denote the pivot in Δ by h and distinguish between the following three cases:

- $h \neq i$ and i is after j in δ_{bj}:

$$\begin{array}{ll} \delta_{aj} = (\ldots, \hat{h}, j, \ldots, i, \ldots) & \delta_{aj}^{i,j} = (\ldots, \hat{h}, i, \ldots, j, \ldots) \\ \delta_{bj} = (\ldots, j, \hat{h}, \ldots, i, \ldots) & \delta_{bj}^{i,j} = (\ldots, \hat{\imath}, h, \ldots, j, \ldots) \end{array}$$

  Of course, h is also the pivot in $\delta_{aj}^{i,j}$. If h were the pivot in $\delta_{bj}^{i,j}$, then $\delta_{aj}^{i,j}$ and $\delta_{bj}^{i,j}$ would be element of the same chain $\Lambda \in \bar{P}(i)$. But then $g^{ij}(\Lambda) = \Delta$, which is impossible by assumption. Since $M_i > M_j$, player i must be the pivot in $\delta_{bj}^{i,j}$. The chain to which $\delta_{bj}^{i,j}$ belongs cannot be an image under f^{ij}, since it is obtained by switching i and j in a permutation in which j is not the pivot. Furthermore, two different starting chains $\Delta, \tilde{\Delta} \in \bar{P}(j)$ cannot give rise to one single chain containing both $\delta_{bj}^{i,j}$ and $\tilde{\delta}_{bj}^{i,j}$, because both permutations are of type $PP-$ or $-P-$ and there can be only one such permutation in a chain.

- $h \neq i$ and i is before j in δ_{bj}:

$$\begin{array}{ll} \delta_{aj} = (\ldots, i, \ldots, \hat{h}, j, \ldots) & \delta_{aj}^{i,j} = (\ldots, j, \ldots, h, \hat{\imath}, \ldots) \\ \delta_{bj} = (\ldots, i, \ldots, j, \hat{h}, \ldots) & \delta_{bj}^{i,j} = (\ldots, j, \ldots, i, \hat{h}, \ldots) \end{array}$$

  Again, it easily follows that h is pivot in $\delta_{bj}^{i,j}$ and by the same argument as in the first case, i must be pivot in $\delta_{aj}^{i,j}$. Also, the chain to which $\delta_{aj}^{i,j}$ belongs cannot be an image under f^{ij} and two different starting chains $\Delta, \tilde{\Delta} \in \bar{P}(j)$ cannot give rise to one single chain containing both $\delta_{aj}^{i,j}$ and $\tilde{\delta}_{aj}^{i,j}$, because both permutations are of type $-PP$ or $-P-$. Moreover, the chains constructed in this second case, containing $\delta_{aj}^{i,j}$, must differ from the chains constructed in the first case, containing $\delta_{bj}^{i,j}$, as a result of the relative positions of h and j.

- $h = i$:

$$\begin{array}{ll} \delta_{aj} = (\ldots, \hat{\imath}, j, \ldots) & \delta_{aj}^{i,j} = (\ldots, j, \hat{\imath}, \ldots) \\ \delta_{bj} = (\ldots, j, \hat{\imath}, \ldots) & \delta_{bj}^{i,j} = (\ldots, \hat{\imath}, j, \ldots) \end{array}$$

Obviously, i is the pivot in both $\delta_{aj}^{i,j}$ and $\delta_{bj}^{i,j}$. So, these two permutations belong to the same chain $\Lambda \in P(i)$. Again Λ cannot be an image under f^{ij}, and since $\Lambda = \Delta$, different starting chains give rise to different Λ's. Finally, the chains constructed in this case must differ from the chains in the first two cases, because the new ones are elements of $\bar{P}(j)$, whereas the chains constructed in the first two cases are elements of $\mathcal{C} \backslash C(j)$.

Combining the three cases, for every element of $\bar{P}(j)$ that is not an image under g^{ij} of any chain in $\bar{P}(i)$, we have found a different element of $P(i)$ that is not an image under f^{ij} of any chain in $P(j)$. Together with Lemma 11.4.4, we have $|P(j)| + |\bar{P}(j)| \leq |P(i)| + |\bar{P}(i)|$.
Similarly, by taking $\Lambda \in P(i)$ such that there exists no $\Delta \in P(j)$ with $\Lambda = f^{ij}(\Delta)$, one can prove the reverse inequality of (11.5). Combining the two inequalities, we conclude $|C(i)| = |C(j)|$. □

Using the previous proposition, we can compute the total weight of all larginals.

Proposition 11.4.6 $\sum_{\sigma \in \Pi(N)} w(\sigma) = (n-1)! \sum_{i \in N} M_i$.

Proof: First note that for every chain $\Gamma \in \mathcal{C}$, we have $|\Gamma| = |\{j \in N \mid \Gamma \in C(j)\}|$. As a result of Proposition 11.4.5, we have $\sum_{\Gamma \in \mathcal{C}} |\{j \in N \mid \Gamma \in C(j)\}| = n|C(i)|$ for all $i \in N$. But then, since $\sum_{\Gamma \in \mathcal{C}} |\Gamma| = n!$, we conclude that $|C(i)| = \frac{n!}{n} = (n-1)!$ for all $i \in N$, so each player belongs to $(n-1)!$ chains. Then the statement immediately follows from Lemma 11.4.2. □

Step 4: numerator, first player

Now we turn our attention to the numerator of (11.2). For this, we partition the set of chains into subsets with the same starting player:

$$\mathcal{C}_k = \{\{\sigma_1, \ldots, \sigma_q\} \in \mathcal{C} \mid \sigma_1(1) = k\}.$$

Note that since player k is by definition never the pivot in σ_1, he is also the first player in $\sigma_2, \ldots, \sigma_q$. It is easily verified that $\{\mathcal{C}_k\}_{k \in N}$ is indeed a partition of $\mathcal{C}$.

For a chain $\Gamma = \{\sigma_1, \ldots, \sigma_q\} \in \mathcal{C}$, we define L^Γ to be the weighted sum of its corresponding larginals:

$$L^\Gamma = \sum_{k=1}^{q} w(\sigma_k)\ell^{\sigma_k}.$$

We compute the numerator in (11.2) by combining the permutations that belong to the same $\mathcal{C}_k$, $k \in N$, deriving an expression for $\sum_{\Gamma \in \mathcal{C}_k} L_i^\Gamma$ for each player $i \in N$. In this step, we consider the special case where $i = k$, while in the next step we compute the payoff to the other players.

Lemma 11.4.7 *For all $i \in N$, $\sum_{\Gamma \in \mathcal{C}_i} L_i^\Gamma = (n-2)! M_i \sum_{j \in N \setminus \{i\}} M_j$.*

Proof: In a similar way as in Proposition 11.4.5, one can show that $|\mathcal{C}_i \cap C(j)| = |\mathcal{C}_i \cap C(k)|$ for all $j, k \in N \setminus \{i\}$. Analogous to Proposition 11.4.6, we then have $\sum_{\sigma \in \Pi(N):\sigma(1)=i} w(\sigma) = (n-2)! \sum_{j \in N \setminus \{i\}} M_j$. Since player i always gets M_i at the first position, the statement follows. □

Step 5: numerator, other players

In this step, we finish the expression for the numerator in (11.2) by computing $\sum_{\Gamma \in \mathcal{C}_k} L_i^\Gamma$ for all $i \in N, i \neq k$. First, in a similar way as in Lemma 11.4.2, one can compute the total weighted larginal for each chain, as is done in the next lemma.

Lemma 11.4.8 *Let $\Gamma = \{\sigma_1, \ldots, \sigma_q\} \in P(i)$. Then for $j = \sigma_1(s)$ we have*

$$L_j^\Gamma = \begin{cases} w(\Gamma) M_j & \text{if } s < p_{\sigma_1}, \\ (v(N) - \sum_{k=1}^{p_{\sigma_1}-1} M_{\sigma_1(k)}) M_j & \text{if } j = i, \\ (v(N) - \sum_{k=1, k \neq p_{\sigma_1}}^{s-1} M_{\sigma_1(k)} + \sum_{k=s+1}^{p_{\sigma_1}+q-1} M_{\sigma_1(k)}) M_j & \text{if } \Gamma \in \bar{P}(j), \\ 0 & \text{if } s > p_{\sigma_1} + q - 1. \end{cases}$$

Example 11.4.4 Of course, $L_1^\Gamma = w(\Gamma) M_1 = 11 \cdot 5 = 55$ and $L_5^\Gamma = 0$. For player 2, the pivot, we have

$$\begin{aligned} L_2^\Gamma &= w(\sigma_1)(v(N) - M_1) + w(\sigma_2)(v(N) - M_1 - M_3) \\ &\quad + w(\sigma_3)(v(N) - M_1 - M_3 - M_4) \\ &= 3 \cdot (10-5) + 4 \cdot (10-5-1) + 4 \cdot (10-5-1-3) \\ &= 35. \end{aligned}$$

Indeed, this equals $(v(N) - \sum_{k=1}^{p_{\sigma_1}-1} M_{\sigma_1(k)})M_2 = (10 - 5) \cdot 7$, as stated in Lemma 11.4.8.

For player 3, who belongs to Γ but is not the pivot, we have

$$\begin{aligned} L_3^{\Gamma} &= w(\sigma_1) \cdot 0 + w(\sigma_2)M_3 + w(\sigma_3)M_3 \\ &= 0 + 4 \cdot 1 + 4 \cdot 1 \\ &= 8, \end{aligned}$$

which equals the expression in Lemma 11.4.8. For player 4, the computation is similar. $\triangleleft$

Lemma 11.4.9 *For all $i, k \in N, i \neq k$, we have $\sum_{\Gamma \in \mathcal{C}_k} L_i^{\Gamma} = (n-2)!(v(N)-M_k)M_i$.*

Proof: We prove the assertion using an iterative procedure, varying the utopia payoffs while keeping $v(N)$ constant. We denote the utopia vector in iteration t by M^t and throughout the procedure, this vector satisfies all our assumptions. We first show that the statement holds for $M^1 = (v(N), \ldots, v(N)) \geq M$. Then we iteratively reduce the components of the utopia vector one by one until we, after finitely many steps, end up in M. For every M^t, we show that for the corresponding (induced) set of chains, the total weighted payoff to i, $\sum_{\Gamma \in \mathcal{C}_k} L_i^{\Gamma,t}$, equals $(n-2)!(v(N) - M_k^t)M_i^t$.

Step 1
Take $M^1 = (v(N), \ldots, v(N))$. Then all the chains consist of one permutation, in which the second player is the pivot. Player i gets 0 if he is after the pivot and $v(N) - M_k^1$ if he is the pivot. There are $(n-2)!$ chains in $\mathcal{C}_k$ in which the latter occurs, each having weight M_i^1. Hence, $\sum_{\Gamma \in \mathcal{C}_k} L_i^{\Gamma,1} = (n-2)!(v(N) - M_k^1)M_i^1$.

Step t
Suppose the statement holds for utopia vector M^{t-1}. If $M^{t-1} = M$, then we are finished. Otherwise, there exists a $j \in N$ such that $M_j^{t-1} > M_j$. We now reduce j's utopia payoff until one of the chains changes, or until M_j is reached.
A chain changes if in one of its permutations, the pivot changes. Obviously, this can only happen if player j is the pivot or before the pivot. Because in the first

permutation of each chain the gap between what the pivot gets and his utopia payoff is smallest, this permutation is the first to change. Denoting this gap corresponding to $\sigma \in \Pi(N)$ by $\gamma(\sigma)$, ie,

$$\gamma(\sigma) = M^{t-1}_{\sigma(p_\sigma)} - (v(N) - \sum_{k=1}^{p_\sigma - 1} M^{t-1}_{\sigma(k)}),$$

the first chain changes when j's utopia payoff is decreased by

$$\gamma = \min\{\gamma(\sigma_1) \,|\, \{\sigma_1, \ldots, \sigma_q\} \in \mathcal{C}_k, \sigma_1^{-1}(j) \leq p_{\sigma_1}\}. \tag{11.6}$$

Assume *for the moment* that the corresponding argument contains one element and denote its first permutation by $\hat{\sigma}$.

If $\gamma \geq M^{t-1}_j - M_j$, then decreasing j's utopia payoff from M^{t-1}_j to M_j does not result in any change in the chains. In this case, the statement holds for M^t_j defined by $M^t_j = M_j, M^t_h = M^{t-1}_h$ for all $h \in N\backslash\{j\}$ and we proceed to step $t+1$.

Otherwise, take $M^t_h = M^{t-1}_h$ for all $h \in N\backslash\{j\}$ and $M^t_j = M^{t-1}_j - (\gamma + \varepsilon)$, where $\varepsilon > 0$ is chosen small enough such that $\hat{\sigma}$ is the *only* permutation in which the pivot changes. In particular, the pivot remains the same in the second permutation of the same chain and in the first permutations of all the other chains.

As mentioned before, $\hat{\sigma}$ is the first in a chain, say $\Gamma \in \mathcal{C}_k$. So, $\hat{\sigma}$ must be either of type $-P-$ or $-PP$. Define $s = \hat{\sigma}^{-1}(i)$ and distinguish between the two cases:

- $\hat{\sigma}$ is of type $-P-$:
  $\hat{\sigma}^R$ is part of another chain, say $\Delta \in \mathcal{C}_k$ with length q. Then the players $\hat{\sigma}(p_{\hat{\sigma}} - q + 1), \ldots, \hat{\sigma}(p_{\hat{\sigma}} - 1)$ and $\hat{\sigma}(p_{\hat{\sigma}} + 1)$ belong to Δ. When the pivot changes in $\hat{\sigma}$, this permutation joins Δ, as type $PP-$, forming chain $\Delta \cup \{\hat{\sigma}\}$. Hence, chain $\Gamma = \{\hat{\sigma}\}$ disappears and the length of Δ is increased by one, while the other chains are not affected. So, it suffices to show that $L^{\Gamma,t-1}_i + L^{\Delta,t-1}_i$ as function of M^{t-1} equals $L^{\Delta\cup\{\hat{\sigma}\},t}_i$ as function of M^t. Using Lemma 11.4.8, we have:

  - $1 < s < p_{\hat{\sigma}} - q + 1$:

$$\begin{aligned} L^{\Gamma,t-1}_i &= M^{t-1}_{\hat{\sigma}(p_{\hat{\sigma}})} M^{t-1}_i \text{ (i is before Γ)}, \\ L^{\Delta,t-1}_i &= (M^{t-1}_{\hat{\sigma}(p_{\hat{\sigma}}+1)} + \sum_{\ell = p_{\hat{\sigma}} - q + 1}^{p_{\hat{\sigma}} - 1} M^{t-1}_{\hat{\sigma}(\ell)}) M^{t-1}_i \text{ (i is before Δ)}, \\ L^{\Delta\cup\{\hat{\sigma}\},t}_i &= (\sum_{\ell = p_{\hat{\sigma}} - q + 1}^{p_{\hat{\sigma}} + 1} M^t_{\hat{\sigma}(\ell)}) M^t_i \text{ (i is before $\Delta \cup \{\hat{\sigma}\}$)}. \end{aligned}$$

– $s = p_{\hat{\sigma}}$:

$$\begin{aligned}
L_i^{\Gamma,t-1} &= (v(N) - \sum_{\ell=1}^{p_{\hat{\sigma}}-1} M_{\hat{\sigma}(\ell)}^{t-1}) M_i^{t-1} \ (\Gamma \in P(i)), \\
L_i^{\Delta,t-1} &= 0 \ (i \text{ is after } \Delta), \\
L_i^{\Delta\cup\{\hat{\sigma}\},t} &= (v(N) - \sum_{\ell=1}^{p_{\hat{\sigma}}-1} M_{\hat{\sigma}(\ell)}^{t}) M_i^{t} \ (i \text{ is “last” in } \Delta \cup \{\hat{\sigma}\}).
\end{aligned}$$

– $p_{\hat{\sigma}} - q + 1 \leq s < p_{\hat{\sigma}}$:

$$\begin{aligned}
L_i^{\Gamma,t-1} &= M_{\hat{\sigma}(p_{\hat{\sigma}})}^{t-1} M_i^{t-1} \ (i \text{ is before } \Gamma), \\
L_i^{\Delta,t-1} &= (v(N) - \sum_{\ell=1}^{s-1} M_{\hat{\sigma}(\ell)}^{t-1} + \sum_{\ell=s+1}^{p_{\hat{\sigma}}-1} M_{\hat{\sigma}(\ell)}^{t-1}) M_i^{t-1} \ (\Delta \in \bar{P}(i)), \\
L_i^{\Delta\cup\{\hat{\sigma}\},t} &= (v(N) - \sum_{\ell=1}^{s-1} M_{\hat{\sigma}(\ell)}^{t} + \sum_{\ell=s+1}^{p_{\hat{\sigma}}} M_{\hat{\sigma}(\ell)}^{t}) M_i^{t} \ (\Delta \cup \{\hat{\sigma}\} \in \bar{P}(i)).
\end{aligned}$$

– $s = p_{\hat{\sigma}} + 1$:

$$\begin{aligned}
L_i^{\Gamma,t-1} &= 0 \ (i \text{ is after } \Gamma), \\
L_i^{\Delta,t-1} &= (v(N) - \sum_{\ell=1}^{p_{\hat{\sigma}}-q} M_{\hat{\sigma}(\ell)}^{t-1}) M_i^{t-1} \ (\Delta \in P(i)), \\
L_i^{\Delta\cup\{\hat{\sigma}\},t} &= (v(N) - \sum_{\ell=1}^{p_{\hat{\sigma}}-q} M_{\hat{\sigma}(\ell)}^{t}) M_i^{t} \ (\Delta \cup \{\hat{\sigma}\} \in P(i)).
\end{aligned}$$

– $s > p_{\hat{\sigma}} + 1$:

$$L_i^{\Gamma,t-1} = L_i^{\Delta,t-1} = L_i^{\Delta\cup\{\hat{\sigma}\},t} = 0 \ (i \text{ is after all three chains}).$$

It is readily checked that in all cases, $L_i^{\Gamma,t-1} + L_i^{\Delta,t-1}$ as function of M^{t-1} equals $L_i^{\Delta\cup\{\hat{\sigma}\},t}$ as function of M^t.

- $\hat{\sigma}$ is $-PP$:
  $\hat{\sigma}^R$ belongs to the same chain as $\hat{\sigma}$. When the pivot changes in $\hat{\sigma}$, this permutation will form a new chain of length one. In the same manner as in the previous case, we can show that the total weighted payoff to i as function of the utopia vector in these two chains remains the same.

So, from these two cases, we conclude $\sum_{\Gamma\in\mathcal{C}_k} L_i^{\Gamma,t} = (n-2)!(v(N)-M_k^t)M_i^t$. Proceed to step $t+1$.

We assumed that the minimal gap in (11.6) is obtained for a unique permutation, $\hat{\sigma}$. Suppose now that there exists another permutation, $\tilde{\sigma}$, with this minimal gap. Because the utopia payoffs are assumed to be strictly positive, $\tilde{\sigma}$ cannot be in the same chain (Γ) as $\hat{\sigma}$, but must be the first permutation of another chain ($\tilde{\Gamma}$). It readily follows from the construction that also the two corresponding "neighbouring" chains Δ and $\tilde{\Delta}$ are different, and moreover, they differ from Γ and $\tilde{\Gamma}$. Hence, we can consider the analysis in step t for $\hat{\sigma}$ and $\tilde{\sigma}$ separately to prove the statement. Finally, our procedure stops after finitely many steps, because in all the changes, the pivot concerned moves towards the back of a permutation. □

Step 6: final

In this final step, we combine our previous results to prove the main theorem.

Proof of Theorem 11.3.1: Let $i \in N$. Then applying Lemmas 11.4.7 and 11.4.9 yields

$$\begin{aligned}
\sum_{\sigma\in\Pi(N)} w(\sigma)\ell_i^\sigma &= \sum_{\Gamma\in\mathcal{C}} L_i^\Gamma \\
&= \sum_{k\in N\setminus\{i\}} \sum_{\Gamma\in\mathcal{C}_k} L_i^\Gamma + \sum_{\Gamma\in\mathcal{C}_i} L_i^\Gamma \\
&= \sum_{k\in N\setminus\{i\}} (n-2)!(v(N)-M_k)M_i + (n-2)!M_i \sum_{k\in N\setminus\{i\}} M_k \\
&= (n-1)!v(N)M_i.
\end{aligned}$$

Then, using Proposition 11.4.6, we have

$$\begin{aligned}\zeta_i &= \frac{\sum_{\sigma\in\Pi(N)} w(\sigma)\ell_i^\sigma}{\sum_{\sigma\in\Pi(N)} w(\sigma)}\\ &= \frac{(n-1)!v(N)M_i}{(n-1)!\sum_{j\in N} M_j}\\ &= \frac{v(N)}{\sum_{j\in N} M_j}M_i.\end{aligned}$$

Hence, $\tau^* = \zeta$. $\square$

As stated in section 2, for the class of compromise admissible games in which, after normalising such that the minimal rights vector equals zero, each player's utopia payoff is at most the value of the grand coalition, the τ^* value coincides with the compromise value. As a result, Theorem 11.3.1 gives a geometric characterisation of the latter on this class of games.

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Samenwerking en verdeling

Samenvatting

Dit proefschrift behandelt een aantal onderwerpen uit de coöperatieve speltheorie. Deze stroming binnen de economische wetenschappen heeft als doel om economische samenwerking formeel wiskundig te modelleren (als een zogeheten spel) om zo tot een "eerlijke" verdeling van de vruchten van die samenwerking te komen. Het is duidelijk dat de vraag of en hoe economische agenten (bijvoorbeeld consumenten of bedrijven) samenwerken niet los gezien kan worden van de vraag wat die samenwerking uiteindelijk voor iedereen oplevert: in beide richtingen beïnvloeden de twee analyses elkaar.

De meest populaire tak van sport binnen de coöperatieve theorie is het zogenaamde *transferable utility* (TU) model. Dit eenvoudige model heeft als centrale eigenschap dat op het moment dat er iets verdeeld moet worden, de betrokken agenten de te verdelen objecten gelijk waarderen. Dit is alleszins een redelijke aanname als de te verdelen pot alleen maar bestaat uit geld en de meeste TU-literatuur richt zich dan ook op situaties waarin de vruchten van samenwerking op een door alle betrokkenen aanvaarde manier uitgedrukt kunnen worden in geld.

Wanneer het fysieke goederen betreft die verdeeld moeten worden, of abstracte goederen zoals bepaalde rechten, kan de waardering voor deze goederen door de verschillende agenten uiteenlopen, en is het TU-model niet toepasbaar. Om dit te ondervangen is een wat rijker model ontwikkeld dat uitgaat van *nontransferable utility* (NTU).

Hoofdstuk 3 behandelt het belangrijke begrip convexiteit. Kortweg heet een spel convex als samenwerken met een grotere groep meer extra opbrengsten genereert dan samenwerken met een kleinere groep. Als gevolg hiervan wil in een convex spel iedereen met elkaar samenwerken, wat de analyse van zulke spelen vereenvoudigt. Convexiteit voor TU spelen is al intensief onderzocht en heeft geleid tot veel be-

langrijke resultaten. Voor NTU ligt de zaak wat gecompliceerder. De literatuur biedt enkele uitbreidingen van de convexiteitsanalyse naar deze context, maar deze zijn lastig te interpreteren en toe te passen. De bijdrage van dit proefschrift is een systematische analyse van de verschillende vormen van convexiteit in het geval van NTU.

Een gedeelte van de voorgaande analyse wordt toegepast in Hoofdstuk 4, waar het begrip monotoniciteit centraal staat. Monotoniciteit is een breed begrip en komt erop neer dat als er in een coöperatieve situatie een verandering optreedt zodat de groep als geheel er beter op wordt, iedere agent apart daarvan ook moet kunnen profiteren. Het eerste deel van het hoofdstuk behandelt deze problematiek in zijn algemeenheid (in termen van een zogenaamde *pmas*) voor het NTU-geval. Het tweede deel behandelt machinevolgordeproblemen. Stel hierbij een machine voor, die een aantal taken moet uitvoeren die beheerd worden door verschillende spelers. Deze spelers staan te wachten totdat zij aan de beurt zijn en willen zo snel mogelijk geholpen worden. Omdat niet alle taken even urgent zijn, kunnen er kosten bespaard worden door toe te staan dat bepaalde spelers van plaats wisselen, waarbij de benadeelden natuurlijk voldoende gecompenseerd worden. In dit kader wordt de zogenaamde uitvalmonotoniciteit geïntroduceerd en onderzocht: op het moment dat een bepaalde speler de wachtrij verlaat, zal de zaak zo geregeld moeten worden dat iedere andere speler erop vooruit gaat, met daarbij de extra eis dat er niet ineens een groepje spelers geprikkeld wordt om de samenwerking op te zeggen. Voor eenvoudige gevallen blijkt dit altijd bewerkstelligd te kunnen worden, maar bij wat complexere kostenstructuren blijkt dit niet altijd mogelijk.

Hoofdstuk 5 gaat over coöperatieve spelen waarin de communicatie tussen de verschillende spelers onderhevig is aan restricties. Hierdoor kunnen bepaalde groepen niet optimaal samenwerken en zal daar bij de modellering rekening mee moeten worden gehouden. De belangrijkste vraag die hier wordt beantwoord heeft betrekking op overerving: in hoeverre worden de eigenschappen (zoals convexiteit) waaraan een bepaalde klasse van NTU spelen voldoet beïnvloed door de communicatierestricties en waar moeten die restricties aan voldoen zodat het voor de verdere analyse geen wezenlijk verschil maakt?

In de standaard coöperatieve theorie richt de analyse zich op de gevolgen van samenwerking voor degenen die daadwerkelijk samenwerken. De gevolgen voor degenen die besluiten om niet mee te doen worden hierbij buiten beschouwing gelaten met de gedachte dat die het verdelingsvraagstuk binnen de groep niet beïnvloeden.

Een effect dat hierbij niet wordt meegenomen is dat deze zogenoemde *spillovers* wezenlijke prikkels kunnen vormen voor spelers in de beslissing om al dan niet mee te doen. Dit strategische aspect staat centraal in het spillovermodel, dat gepresenteerd wordt in Hoofdstuk 6. Dit nieuwe model wordt geïllustreerd aan de hand van een coalitieformatieprobleem en een netwerkprobleem.

Hoofdstuk 7 is het eerste in een reeks van vier hoofdstukken die in het teken staan van bankroetproblemen. Bankroetprobleem is een algemene (ietwat misleidende) benaming voor het meest eenvoudige type verdeelprobleem: er is een hoeveelheid geld beschikbaar die verdeeld moet worden onder een aantal spelers, die allen een gegeven claim hebben op dit bedrag – denk hierbij aan een faillissement, waarbij de restwaarde van het geliquideerde bedrijf onder de schuldeisers moet worden verdeeld. In het algemeen zal er meer geclaimd worden dan er beschikbaar is, zodat er allerlei criteria bedacht moeten worden op basis waarvan het geld eerlijk verdeeld kan worden. De bijdrage in dit hoofdstuk heeft betrekking op referenties: naast de gebruikelijke claims is er nog een apart referentiekader (bijvoorbeeld ten gevolge van wet- of regelgeving) dat bij de verdeling in ogenschouw dient te worden genomen. De voorgestelde oplossingsmethode is gebaseerd op een compromis-principe.

In Hoofdstuk 8 wordt een substantiële uitbreiding op het bankroetmodel gepresenteerd. In plaats van een enkele claim heeft iedere speler een heel scala aan claims die gedifferentieerd zijn naar verschillende *issues*. Bij een failissement kan gedacht worden aan claims op basis van openstaande obligaties, leverancierskrediet, opties of andere financiële verplichtingen. Bij het verdelingsprobleem kan vervolgens rekening worden gehouden met de aard van de verschillende claims. Voor het resulterende model wordt een oplossing voorgesteld die gebruik maakt van het principe van consistentie, wat erop neerkomt dat grote problemen in essentie op dezelfde wijze opgelost dienen te worden als kleinere problemen. In Hoofdstuk 9 wordt een andere kijk op dit multi-issue model gegeven, gebaseerd op een alternatief consistentie-idee.

In veel economische situaties kunnen de participerende spelers op natuurlijke wijze onderverdeeld worden in onderling disjuncte groepen van spelers met dezelfde karakteristieken. Hoofdstuk 10 behandelt bankroetsituaties waarin dit het geval is. Aan de hand van een bekend oplossingsconcept voor standaard bankroetproblemen wordt een aantal methoden gepresenteerd om met deze *a priori* groeperingen rekening te houden.

Dit proefschrift besluit met Hoofdstuk 11, waarin een meetkundige interpretatie van een veelgebruikte compromis-oplossingsmethode centraal staat. Dit verrassende theoretische resultaat heeft technisch heel wat voeten in de aarde en het overgrote deel van dit hoofdstuk is dan ook gewijd aan het bewijs hiervan.

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